On frequency distribution of impulsive feedback control times

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Received 30 October 2017; received in revised form 14 April 2018; accepted 16 June 2018
Available online 7 July 2018

Abstract

The present work considers an impulsive feedback control problem governed by a system of ODEs defined on a class of isolated time scales. The control is activated once the unique solution’s trajectory leaves a bounded connected domain. A periodicity of the controlled system’s solution in the context of shift-invariant time scales are discussed. Making use of the $\Delta$-measure theory, a cumulative distribution of the control times is introduced. Then with the help of dynamic inequalities on time scales such as Opial-type inequality, upper estimations for the occurrence frequency of the control times are obtained. A uniform distribution of the control times mod 1 is studied and a necessary condition for the uniformity of distribution of control times in terms of critical dynamic of the controlled system is obtained.

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1. Introduction

Suppose a non-controlled dynamic is governed by an initial value problem

$$
\begin{align*}
\dot{x} &= f(s, x(s)), \quad s \in \mathbb{R}^+ \\
x(0) &= x_0 \in \Omega,
\end{align*}
$$

with $f : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying existence and uniqueness conditions and $\Omega \subset \mathbb{R}^N$ is a connected bounded domain. The dimension $N$ refers to the depth of our investigation which is reflected in the model. An impulsive (instantaneous) state-dependant control $u$ is performed

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https://doi.org/10.1016/j.jfranklin.2018.06.037

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to keep the state within the domain $\Omega$ and it is defined as
\[ u(s) = g(x(s))\delta(H(x(s))) , \]
where $g : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is continuous and $H(x) = 0$ defines an $(N - 1)$-dimensional manifold $\partial \Omega$.

For the controlled system
\[
\begin{cases}
\dot{x} = f(s, x(s)) + g(x(s))\delta(H(x(s))), & s \in \mathbb{R}^+ \\
x(0) = x_0 \in \Omega,
\end{cases}
\]
we have a difference equation which is called a universal map introduced as
\[ x(n + 1) = x(n) + \omega(n) + g(x(n)), \]
with $x(n) = x(t_n^-)$, $\{t_n\}$ as a sequence of times at which the impulsive control is performed, and $\omega(n) = \int_{t_n}^{t_{n+1}} f(\tau, x(\tau))d\tau$. Apparently, $\{x(n)\} \subset \partial \Omega$. See [16] for the background of the controlled system (1.3) and Section 2.2, [19] for derivation of the universal map.

A great deal of research have been conducted on impulsive control systems in theory and real world applications (see e.g., [16–18,20]). Seshadri and Sujith [20] utilized controls introduced as Eq. (1.2) applied to a class of autonomous systems to investigate a chaos formation in a turbulent combustion system. Chaos anti-control using an impulsive control of type Eq. (1.2) applied to a class of Hamiltonian systems is studied by Nategh et al. [17] whereupon a secure Li-Yorke chaotic encryption scheme is introduced. Schaub and Alfried [18] addressed a control of spacecraft formation flying where an impulsive approach was utilized. In spite of an admissible increase in a fuel cost, the proposed method illustrated a numerical preference over optimal control methods.

For the controlled system (1.3), the special case $\Omega = B(x_0, r)$, for some positive number $r$ is considered in [16] and the following results are obtained

(I) Suppose $g$ is non-zero on $\partial \Omega$ then $\{t_n\}$ is finite or unbounded.

(II) Suppose we replace the ordinary differential operator in (1.1) with $\mathcal{L} = \sum_{j=1}^{m} a_j \frac{d^j}{dt^j}$ and consider the one-dimensional problem
\[
\begin{cases}
\mathcal{L} x(s) = f(s, x(s)) + g_0\delta(x(s) - x_0 - r) + h_0\delta(x(s) - x_0 + r), & s \in \mathbb{R}^+ \\
x(0) = x_0 \in \Omega.
\end{cases}
\]

Suppose $f \in L(0, \infty)$ and $\{t_n\}$ is infinite and convergent. Then $x(t)$ hits both lines $y = x_0 - r$ and $y = x_0 + r$ infinitely many times (the boundary $\partial B(x_0, r)$ consists of two point).

Property I states that in case $g$ is not neutral to the boundary, the control may be feasible since the control times are not accumulated on a specific time, otherwise, the controller fails to perform control action infinitely many times in a finite interval.

According to property II, if the control times $\{t_n\}$ are convergent, which signals the impossibility of control due to the accumulation problem mentioned above, then there must be two targets for which the trajectory has a tendency. In other words with the opposite deduction, in case the trajectory $x(t)$ has only one natural tendency to either of the boundary limits, i.e., either of the lines $y = x_0 - r$ or $y = x_0 + r$, then the control process may be feasible since the time sequence will be distributed divergently on the half axis.

The aim of this work is to present a discrete counterpart of the control problem (1.3) in the sense of an isolated time scale and then to investigate distributional properties of the
control times. Hence the achievements in [16] will be extended and the results obtained in this paper will provide a deeper understanding of distributions of control times as well as critical dynamic of the controlled system.

Time scales are defined as closed subsets of the real line which inherit the same Euclidean topology. The calculus on time scales has been a break-through in unification of difference and differential calculus. For an unfamiliar reader, the fast review paper [1] is recommended.

Periodic and almost periodic functions on time scales is rather a new topic which has been reported in [14,22–24,26]. In spite of continuous time systems, the concept of periodicity or almost periodicity is linked to the periodicity of the time scale. It simply means that for a positive number $T$, we need a shift invariance property of magnitude $T$, i.e., for a time scale $\mathbb{S}$

$$T \pm \mathbb{S} = \mathbb{S}. \quad (1.6)$$

This has made Kaufmann and Raffoul [14] to discuss the concept of a periodic time scale.

After a time scale re-formulation of the main problem in Section 2, we discuss the periodicity of the controlled solution. It is noteworthy that, the contents of the discussions in Section 2 are independent of the notions and definitions in [22]. However an interested reader may find a similarity of Lemma 2.4 and the shift-invariance of the graininess function $\mu$ discussed in [22].

The Section 3 is devoted to the main results of this paper which is investigating on distributional properties of the sequence of control times and their relevances with critical states. Here the word “critical” refers to those states of the controlled system’s solution denoted by $x(t)$ that they were subject to jump out of the bounded domain before implementation of the control. i.e., given an initial time $t$, the successor of a critical time is the first control time greater than $t$ denoted by $h(t)$. Section 3 begins with a formulation of $h(t)$ in terms of $t$, $\Omega$ and the non-linearity.

From a governing equation for critical dynamic presented in Theorem 3.1 given by Eq. (3.5) we deduce scattering patterns of control times. The Eq. (3.5) in special cases also imply a class of ODEs which govern critical dynamic stated in Proposition 3.4.

In order to define a cumulative frequency distribution of a given subset of an isolated time scale we employ the $\Delta$-measure theory. The $\Delta$-Lebesgue measure is a time scale counterpart of the Lebesgue measure which was constructed by Guseinov [13] where he followed the Carathéodory’s approach. That provides a foundation for an integration theory on an arbitrary closed subsets of the real line (see also [10]).

In Definition 3.5 we introduce a cumulative frequency distribution of $\mathbb{K}$ in $\mathbb{S}$ for isolated time scales $\mathbb{K} \subset \mathbb{S}$. It is explained that why lim sup vs lim should be used. Hence, the Definition 3.5 amends the definition of density denoted by $\delta_\Omega$ (Definition 3.1, [21]) which is based on taking a lim vs lim sup.

Inequalities on time scales have been an applicable topic since 90’s which exhibit beautiful challenging problems [2–4,7,25]. Another achievement of the Section 3 is employing some of these inequalities and the Eq. (3.5) to obtain upper estimations for frequency of the control times.

A modulated uniform distribution of the control times and a critical dynamic have been discussed in Section 3.1. With the help of $\Delta$-integrability in the Riemann-sense, a necessary condition for uniform distribution of the control times in terms of the critical dynamic of the controlled system’s solution is obtained.
2. Time scale re-formulation

It is evident that time has a discrete nature in simulations. Also the discrete property of time might be a consequence of clock resolutions, lack of observability within a short time interval or periodicity in some phenomena. This fact makes us to reformulate the system (1.3) for an isolated sequence of times rather than the continuum.

Time scales are closed subsets of the real line (including \( \mathbb{R} \)) which is meant in this sequel as a set of scattered points denoted by \( \mathbb{S} \) and it is introduced by \( 0 \in \mathbb{S} \), \( \max \mathbb{S} = \infty \). A positive graininess function \( \mu \) is defined by \( \mu(s) = \sigma(s) - s \) where \( \sigma(s) = \inf \{ \tau \mid s \leq \tau \} \) is the step-forward function. An interested reader may refer to [1,9] for preliminaries on the calculus on time scales.

The basic notations of important time scales in this sequel are as follows

\[
\begin{align*}
\mathbb{N} &= \{1, 2, \ldots \}, & \mathbb{Z} &= \{\ldots, -2, -1, 0, 1, 2, \ldots \}, \\
\mathbb{Z}^+ &= \{0, 1, 2, \ldots \}, & \mathbb{R}^+ &= (0, \infty).
\end{align*}
\]

(2.1)

To introduce a discrete counterpart of the impulse, we need Hilger’s Dirac delta function. For definitions and discussions on the distributional properties see [5,11,12]. Making a slight change to the mentioned definitions, we bring the time scale impulse function defined on \( \mathbb{S} \) by

\[
\delta_{\mathbb{S}^C}(x(s)) = \begin{cases} \frac{1}{\rho(s)}, & x^\sigma(s) \in \Omega^C \\ 0, & \text{otherwise} \end{cases}
\]

(2.2)

where \( \rho \) is the step-backward function \( \rho(s) = \sup \{\tau \mid \tau \leq s\} \), \( x^\sigma = x \circ \sigma \) and \( \mu^\rho = \mu \circ \rho \). Now with the notations above, the controlled system takes the form of

\[
\begin{align*}
\begin{cases}
x^\Delta(s) &= f(s, x(s)) + g(x^\sigma(s))\delta_{\mathbb{S}^C}(x(s)), & s \in \mathbb{S}\setminus 0 \\
x(0) &= x_0 \in \Omega.
\end{cases}
\end{align*}
\]

(2.3)

The time scale formulation Eq. (2.3) includes the continuous case Eq. (1.3). For \( \mathbb{S} = \mathbb{R}^+ \cup \{0\} \), we have \( \mu(s) = 0 \) and the Hilger’s Dirac delta function can be replaced by \( \delta(s) \). Indeed the complement of \( \Omega \) in (2.2) is an overmuch condition since the solution of the system would be piecewise continuous so, the control is activated once \( x(s) \) touches the boundary.

2.1. Periodicity and asymptotic periodicity

The aim of this subsection is to show that, periodicity of the solution of the system (2.3) strongly depends on the scattering property of the time scale \( \mathbb{S} \) which is reflected in its graininess function \( \mu \).

Let \( \mathbb{T} = \{t_n\} \subset \mathbb{S} \) stands for the times at which, the trajectory meets \( \Omega^C \). If \( x^* \) and \( x \) denote the solutions to the non-controlled and controlled systems respectively then

\[
x^*(t_1) = x_0 + \int_0^{t_1} f(s, x(s)) \Delta s \in \Omega^C
\]

(2.4)

and

\[
x(t_1) = x_0 + \int_0^{t_1} f(s, x(s)) \Delta s + g(x^*(t_1)) \in \Omega.
\]

(2.5)
Fig. 1. Controlled system’s trajectory $x(s) \ (s \in S)$. The controller would be able to keep the trajectory within $\Omega$ provided the system does not undergo bursting.

It is clear that $x(\rho(t_1)) \in \Omega$. Also $x(t_1)$ is considered as a new initial condition for the controlled system as

$$
\begin{cases}
  x^\Delta(s) = f(s, x(s)) + g(x^\sigma(s)) \delta_{\Omega^C}(x(s)), & s \in (t_1, t_2]_S \\
  x(t_1) = x^*(t_1) + g(x^*(t_1)) \in \Omega,
\end{cases}
$$

where $(t_1, t_2]_S$ is the time scale interval defined as $\{s \in S | t_1 < s \leq t_2\}$. From the notations $x^*(n) = x^*(t_n)$ and $\omega(n) = \int_{t_n}^{t_{n+1}} f(s, x(s)) \Delta s$, for $n \in \mathbb{N}$ we have (see Fig. 1)

$$
x^*(n + 1) = x^*(n) + \omega(n),
$$

$$
x^*(n) = x(n) - g(x^*(n)),
$$

$$
x(s) \in \Omega \quad \text{for} \ s \in S,
$$

$$
x^* = x \quad \text{holds on} \ S \setminus T,
$$

$$
\{x^*(n)\} \subset \Omega^C, \quad \{x(n)\} \subset \Omega.
$$

Now the control is activated in such a way that, the controlled system initiates at $x(n) \in \Omega$ once the system is perturbed by the $n$th impulsive control action. For simplicity, we set $x(n) = x_0$. Then the corresponding universal map is given by

$$
x^*(n + 1) = x_0 + \omega(n) - g(x^*(n)).
$$
Moreover, from the Eq. (2.8) the control law is known as
\[ g(x) = x_0 - x, \quad x_0 \in \Omega, \ x \in \mathbb{R}^N. \] (2.13)

Suppose \( \mu \) is constant, i.e., \( \mathbb{S} = p\mathbb{Z}^+ \) and let \( \Omega = B(x_0, r), \ (p, r \in \mathbb{R}^+) \). For autonomous counterpart of the system (2.3), i.e., \( \frac{df}{ds} = 0 \), the control law (2.13) and the symmetry of the domain implies periodicity:

Since the solution to the controlled system initiates at the same state in each time window \([t_n, t_{n+1})_{\mathbb{S}}\), the uniqueness assumption guarantees the same dynamic. It implies that the control time sequence \( \mathbb{T} \) has a constant graininess function \( \mu_{\mathbb{T}}(t_n) = t_{n+1} - t_n \) that is to say, \( x \) is periodic. For non-autonomous system (2.3) we have a periodic solution if and only if \( f(\cdot, x) \) is periodic \((x \in \mathbb{R}^N)\). The following statement summarizes the discussion above.

**Theorem 2.1.** Let \( \Omega = B(x_0, r), \ \mathbb{S} = p\mathbb{Z}^+ \ (p, r \in \mathbb{R}^+) \) and suppose \( x^* \) is the unique solution to the non-controlled system corresponding to the controlled system (2.3) which satisfies \( \{x^+(s)\} \cap \Omega^C \neq \emptyset \) and let \( x(n) = x_0, \ (n \in \mathbb{N}) \). Then the control law is \( g(x) = x_0 - x \ (x \in \mathbb{R}^N) \). Moreover the controlled system is periodic with period \( T = t_1 \) if and only if \( f(\cdot, x) \) is periodic with period \( T = t_1 \). In case \( \frac{df}{ds} = 0 \), the control law (2.13) is sufficient to make the controlled system (2.3) periodic.

**Remark 2.2.** The same deduction as above with making use of \( \mu_{\mathbb{R}^+} = 0 \) shows that the piece-wise continuous solution of the controlled system (1.3) is periodic provided \( x(t_n^+) = x_0 \in \Omega \) and the system is autonomous. If this is the case then \( \{x(n)\} = \{x(t_n^-)\} \subset \partial \Omega \) and \( g(x) = x_0 - x \) for \( x \in \mathbb{R}^N \).

The setting \( \{x(n)\} = \{x_0\} \) which is provided the periodicity, might be impossible. For example in social dynamics, due to a resistance of the community in returning to its earlier initial standards, it makes sense to apply \( \{x(n)\} = \{x_0 + r_n\} \) with \( \{r_n\} \subset \mathbb{R}^N \) as a vanishing sequence satisfying \( x_0 + r_n \in \Omega \).

For \( y \in \Omega \) let \( x^+(s, y) \) and \( x(s, y) \) denote the non-controlled and controlled solutions initiated at \( y \). We denote \( \mathbb{T} = \{t_i\} \) to be the sequence of control times pertaining to the sequence of initial values \( \{x_0 + r_n\} \). Also \( \mathbb{T}_n \) and \( \mathbb{T}_n \) are defined by \( \{t_j| j \geq n\} \) and \( \{t_j| j \geq n\} \) resp.

**Theorem 2.3.** Let \( g(x) = x_0 - x + r_n, \ x_0 \in \Omega, \ x \in \mathbb{R}^N, \ \{r_n\} \subset \mathbb{R}^N \) be a vanishing sequence, \( \{x_0 + r_n\} \subset \Omega \) and suppose \( \mu_{\mathbb{T}} \) is bounded. If \( x^+(s, x_0) \cap \partial \Omega = \emptyset, \ (s \in \mathbb{S}) \) and \( f(s, \cdot) \) is continuous, then \( \mathbb{T}_n = \mathbb{T}_n \) for \( n \) large enough.

**Proof.** Consider the system (2.3) with an initial time \( t_n \in \mathbb{T} \) for some \( n \in \mathbb{N} \). Let \( t_k^{(n)} (k(n) \in \mathbb{N}) \) be the first control time for which \( x^*(\rho(t_k^{(n)}), x_0 + r_n) \in \Omega \) and \( x^*(t_k^{(n)}), x_0 + r_n) \in \Omega^C \). From the assumptions and

\[
x(t, x_0 + r_n) = r_n + x(t, x_0) + \int_0^t \left[ f(s + t_n, x(s + t_n, x_0 + r_n)) - f(s + t_n, x(s + t_n, x_0)) \right] \Delta s
\]

\[
= r_n + x(t, x_0)
\]

\[
+ \sum_{s=0}^{t} \left[ f(s + t_n, x(s + t_n, x_0 + r_n)) - f(s + t_n, x(s + t_n, x_0)) \right] \mu(s + t_n),
\]

where \( t < \min\{t_{n+1}, t_k^{(n)}\} \), we conclude that \( t_{n+1} = t_k^{(n)} \) provided \( n \) is large enough. Using the same deduction for the intervals \( (t_{n+j}, t_{n+j+1})_{\mathbb{S}}, \ (j \in \mathbb{N}) \) proves the assertion. \( \square \)
The periodicity result in Theorem 2.1 does not necessarily hold for arbitrary isolated time scales since a $t_1$-shift invariance property of $\mathbb{S}$ is needed, that is $t_1 \pm \mathbb{S} = \mathbb{S}$.

**Lemma 2.4.** Suppose $\mathbb{S}$ is $T$-shift invariant for some positive number $T$. Then $\mu$ is $T$-shift invariant in the sense that

$$\mu(s) = \mu(s \pm T), \quad s \in \mathbb{S} \quad (2.14)$$

**Proof.** Since $0 \in \mathbb{S}$ then $T \in \mathbb{S}$. The definition of $\mathbb{S}$ implies that $\mathbb{S} = \{\sigma^n(0) \mid n \in \mathbb{Z}\}$. From $T$-shift invariance, there exist monotone functions $p, q : \mathbb{Z} \to \mathbb{Z}$ so that for $n, m \in \mathbb{Z}$ we have

$$\begin{cases} \sigma^n(0) + T = \sigma^{p(n)}(0), & n < p(n) \\ \sigma^m(0) = \sigma^{q(m)}(0) + T, & q(m) < m \end{cases} \quad (2.15)$$

Therefore the identity below is verified

$$\sigma^{p(n+1)}(0) - \sigma^{p(n)}(0) = \sigma^{n+1}(0) - \sigma^n(0). \quad (2.16)$$

In other words $\mu(\sigma^n(0)) = \sigma^{p(n+1)}(0) - \sigma^{p(n)}(0)$, and it gives rise to

$$\sigma(\sigma^n(0) + T) = \sigma(\sigma^{p(n)}(0))$$

$$= \sigma^{p(n+1)}(0)$$

$$= \sigma^{n+1}(0) + T, \quad (2.17)$$

otherwise there exists $p(n) < m < p(n + 1)$ and $q(m) < n$ for which

$$\sigma^m(0) = \sigma^{q(m)}(0) + T < \sigma^n(0) + T = \sigma^{p(n)}(0), \quad (2.18)$$

that is a contradiction. We conclude that $\sigma(s + T) = \sigma(s) + T$ ($s \in \mathbb{S}$) and it verifies

$$\mu(s) = \sigma(s) - t = \sigma(s) + T - (s + T)$$

$$\quad = \sigma(s + T) - (s + T) = \mu(s + T). \quad (2.19)$$

$\square$

**Theorem 2.5.** Suppose $\Omega = B(x_0, r), x(n) = x_0, (n \in \mathbb{N})$ and $\frac{\partial f}{\partial x} \equiv 0$ or $f(s + t_1, x) = f(s, x)$ ($s \in \mathbb{S}, x \in \mathbb{R}^N$). Then the solution of the controlled system (2.3) is $t_1$-periodic if and only if $\mathbb{S}$ is $t_1$-shift invariant.

**Proof.** From Theorem 2.1 the shift invariance property implies periodicity. Having a periodic solution to the controlled system

$$x_0 + \sum_{s=0}^{\rho(t)} f(x(s)) \mu(s) = x_0 + \sum_{s=0}^{\rho(t)} f(x(s + t_1)) \mu(s + t_1), \quad (2.20)$$

and it gives rise to $\mu(s + t_1) = \mu(s)$ and this completes the proof. $\square$

3. Distribution of the control times $T$

From the formulation of the control problem (2.3) it is evident that for a given control time $t \in \mathbb{T}$, its successor in $\mathbb{T}$ denoted by $h(t)$ satisfies

$$h(t) = \min \left\{ s \in (t, \infty)_{\mathbb{S}} \left| x_t + \int_t^s f(s, x^+(s)) \Delta s \in \Omega^C \right. \right\}, \quad (3.1)$$
where $x^*$ is the solution to the initial value problem $\mathcal{P}_i$ defined by

$$ \mathcal{P}_i : \begin{align*}
\dot{x}(t) &= f(x(t), x(s)), \quad s \in (t, \infty)\mathbb{S} \\
\dot{x}(0) &= x_0.
\end{align*} \tag{3.2} $$

Indeed for each $t \in \mathbb{T}$ there exists a time interval $[t, \rho(h(t))]\mathbb{S}$ and a system (3.2) which governs the dynamic of the controlled system within this interval. However with the help of Eq. (2.9) and setting $x_i = x_0 \ (t \in \mathbb{T})$ one can merge all these initial value problems together by introducing

$$ \tilde{f}(s, x) = \begin{cases} 
\frac{1}{\mu(s)}(x - x_0), & s \in \mathbb{T} \\
\tilde{f}(s, x), & s \in \mathbb{S}\setminus\mathbb{T}
\end{cases} \tag{3.3} $$

Then $x$ satisfies

$$ \begin{align*}
\dot{x}(s) &= \tilde{f}(s, x(s)), \quad s \in (0, \infty)\mathbb{S} \\
\dot{x}(0) &= x(t) = x_0, \quad t \in \mathbb{T}
\end{align*} \tag{3.4} $$

and we have $x(s) \in \Omega, \ (s \in \mathbb{S})$.

Let’s assume that $\Omega = \mathbb{B}(x_0, r)$. It should be noticed that, there exists a critical radius $r_0 = \|x^*(0) - x_0\|$ for which, the controlled system’s solution is a singleton $\{x_0\}$ provided $r \leq r_0$. To exclude the triviality, it is assumed that $r_0 < r$. The next statements, shed a light on critical dynamics of $x$ that is, dynamics of $x(h(t))$ and $x(\rho(h(t)))$ in the special case $\Omega = \mathbb{B}(x_0, r)$.

**Theorem 3.1.** Let $\Omega = \mathbb{B}(x_0, r)$ and suppose $\mathbb{T}$ is unbounded. Then for $t \in \mathbb{T}$ there exists $\lambda(t) \in (0, 1]$ such that $h(t)$ is the minimum $s \in (t, \infty)\mathbb{S}$ which satisfies

$$ \sum_{i=1}^{N} \left( x_{i,j} - x_{0,j} + \int_{t}^{s} f_i(t, x(\tau)) \Delta \tau - \lambda(t) \mu(\rho(s)) f_i(\rho(s), x^\rho(s)) \right)^2 = r^2. \tag{3.5} $$

**Proof.** The Eq. (3.5) comes from the fact that, the line segment $[x^\rho(h(t)), x^*(h(t))]$ intersects $\partial \Omega$ since $x^*(h(t))$ is the first state that lies outside $\Omega$. It implies the existence of a parameter $\lambda = \lambda(t)$ which satisfies $H(\lambda x^\rho(h(t)) + (1 - \lambda)x^*(h(t))) = 0$, where $H(x) = \sum_{i=1}^{N} (x_i - x_{0,i})^2 - r^2$. \hfill $\square$

Although we have considered an isolated time scale but for an arbitrary closed set $\mathbb{S}$, we may observe the following statements.

**Theorem 3.2.** Let $\Omega = \mathbb{B}(x_0, r), \ \{x_i\} = \{x_0\}$ and suppose $\mathbb{S}$ is an arbitrary time scale and the control time sequence $\mathbb{T} \subset \mathbb{S}$ is infinite. Assume that $f(s, x) \neq 0$, for $(s, x) \in \mathbb{S} \times \mathbb{B}(x_0, r)$.

1. If $\{x(\rho(t))\} \subset \Omega$, and $f$ is bounded on $\mathbb{S} \times \mathbb{R}^N$, then $\inf_{\rho(t)} \mu(\rho(t)) > 0$.
2. If $f$ is bounded on $\mathbb{S} \times \mathbb{R}^N$, and $\inf_{\rho(t)} \mu(\rho(t)) = 0$, then there exists a subsequence $\{x(\rho(t))\} \subset \Omega$, so that $x(\rho(t)) \to \partial \Omega$. 
Proof.

(1) From the proof of Theorem 3.1 we conclude the existence of a function $\xi: \mathbb{T} \to (0, 1]$ which satisfies

$$\left\| \xi(t) \mu(\rho(h(t))) f(\rho(h(t)), x(\rho(h(t)))) + x(\rho(h(t))) - x_0 \right\| = r, \quad (3.6)$$

where $\xi(t) x(\rho(h(t))) + (1 - \xi(t)) x(\rho(h(t)))$ lies on the boundary for some $\xi(t) \in (0, 1]$. It is clear that, $x(\rho(h(t))) \rightarrow \partial \Omega$ if and only if $\xi(t) \rightarrow 0$. From the assumption we conclude that $\inf_{t \in \mathbb{T}} \xi(t) > 0$ and therefore $\inf_{t \in \mathbb{T}} \mu(\rho(t)) > 0$ since

$$\frac{r - \|x(\rho(h(t))) - x_0\|}{\xi(t) \left\| f(\rho(h(t)), x(\rho(h(t))) \right\|} \leq \mu(\rho(h(t))), \quad (3.7)$$

as a necessary condition for Eq. (3.6) holds.

(2) If in contrary $\{x(\rho(h(t)))\} \subset \Omega$ holds then $x(h(t)) = \frac{x^*(h(t)) - x(\rho(h(t)))}{\mu(\rho(h(t)))}$ approaches to infinity as $t$ spans a subsequence $\mathbb{T}_1 \subset \mathbb{T}$ that makes $\mu(\rho(h(t)))$ vanish. This contradicts the boundedness of $f$.

Example 3.3. From $\Delta$-differentiating Eq. (3.6) with respect to $s = h(t)$ we obtain

$$\sum_{i=1}^{N} \left( \xi(t) \mu(\rho(s)) f_i(\rho(s), x(\rho(s))) + x_i(\rho(s)) - x_{0,i} \right)$$

$$\times \left( \xi(t) \frac{\mu(\rho(s)) - \mu(\rho(s))}{\mu(\rho(s))} f_i(\rho(s), x(\rho(s))) + f_i(\rho(s), x(\rho(s))) \frac{s - \rho(s)}{\mu(\rho(s))} \right.$$

$$\left. + \xi(t) \mu(\rho(s)) \left\{ \frac{\partial f_i}{\partial s}(\rho(s), x(\rho(s))) \frac{s - \rho(s)}{\mu(\rho(s))} \right. \right.$$ 

$$\left. + \sum_{j=1}^{N} \frac{\partial f_j}{\partial x_j}(\rho(t), x(\rho(t))) f_j(\rho(t), x(\rho(t))) \frac{s - \rho(s)}{\mu(\rho(s))} \right\} = 0. \quad (3.8)$$

Let $f_i(t, x) = tx_i$, $\{x_i\} = \{x_0\}$, $\mathbb{T} = p\mathbb{Z}^+ \ p \in \mathbb{R}^+$ and suppose that $t \geq p$. Then Eq. (3.8) reads

$$\sum_{i=1}^{N} \left( p(s - p)\xi(t)x_i(s - p) + x_i(s - p) - x_{0,i} \right)$$

$$\times \left( (s - p)x_i(s - p) + p\xi(t)(x_i(s - p) + (s - p)^2x_i(s - p)) \right) = 0. \quad (3.9)$$

since $\mu(\rho(s)) = p$ and $\rho(s) = s - p$. After canceling the factor $s - p + p\xi(t)(1 + (s - p)^2)$ which is positive, we obtain

$$\left( 1 + p(s - p)\xi(t) \right) \sum_{i=1}^{N} x_i^2 = \sum_{i=1}^{N} x_{0,i}x_i(s - p), \quad (3.10)$$
and we conclude that

\[
\|x(s - p)\|^2 = \frac{x_0}{1 + p(s - p)\xi(t)} x(s - p),
\]

(3.11)
since \(0 < 1 + p(s - p)\xi(t)\).

It is clear from Eq. (3.11) that, \(\|x(s - p)\| \leq \frac{\|x_0\|}{1 + p(s - p)\xi(t)},\) that is

\[
h(t) \leq \frac{p^2 - 1}{p\xi(t)} + \frac{\|x_0\|}{p\xi(t)\|x(s - p)\|},
\]

(3.12)
provided \(x(s - p) \neq 0). Moreover \(1 < \frac{\|x_0\|}{\|x(s - p)\|},\) for every \(t \in \mathbb{T}.\) Hence, the closure of the critical states \(\{x^0(t)\}\) has a positive distance from the initial state \(\{x_0\}.\)

A more general case is as follows.

**Proposition 3.4.** Let \(\Omega = B(x_0, r)\) and \(\{x_t\} = \{x_0\}.\) Suppose \(\mathbb{S} = p\mathbb{Z}^+\) and \(\frac{\partial f}{\partial x_j}(s, x) = 0,\) for \(i \neq j, i, j = 1, 2, \ldots, N, (s, x) \in \mathbb{S} \times \mathbb{R}^N \setminus \{0\}.\) Then \(x(s)\) for \(s \in \{t - p | t \in \mathbb{T}\}\) satisfies the equation below

\[
x^{\Delta A} \left[(p\xi(t))^2 x^{\Delta} + p\xi(t)(x - x_0) \right] + x^{\Delta} \left[p\xi(t)x^{\Delta} + x - x_0 \right] = 0.
\]

(3.13)

In what follows, a cumulative distribution function will assign a frequency number to a controlled system and that would measure the occurrence of control times in a given time scale. Then we bring an upper estimation for the frequency of \(\mathbb{T}\) in \(\mathbb{S}.\) In view of the concept of \(\Delta\)-Lebesgue measure denoted by \(\mu_\Delta\) discussed in [10,13], we have the definition below.

**Definition 3.5.** Let \(\mathbb{K} \subset \mathbb{S}\) are isolated time scales where \(\min \mathbb{S} = 0,\) \(\max \mathbb{S} = \infty\) and \(\mathbb{K} \neq \emptyset.\) The frequency of \(\mathbb{K}\) in \(\mathbb{S}\) is defined as

\[
\tilde{\mu}_\Delta(\mathbb{K}) = \lim sup_{t \in \mathbb{K}, t \to \infty} \frac{\mu_\Delta(\mathbb{K}_t)}{\mu_\Delta(\mathbb{S}_t)},
\]

(3.14)
where \(\mathbb{S}_t = \{s \in \mathbb{S} | s \leq t\}\) and \(\mathbb{K}_t = \{s \in \mathbb{K} | s \leq t\}.\)

In Eq. (3.14) the lim sup can not be replaced with lim. For example let \(\mathbb{S} = \mathbb{Z}^+\) and \(p \in \text{Prime}.\) From the representation of \(t \in \mathbb{T} \cup \{0\}\) by \(t = pk + \kappa (k \in \mathbb{Z}^+, \kappa = 0, 1, \ldots, p - 1),\) if \(\mu_\Delta(\mathbb{K}_{pk + \kappa}) = k(k + 1)\) then for \(\{\frac{\mu_\Delta(\mathbb{K}_t)}{t}\}\) we have a subsequence limit set \(\{\frac{1}{p}, \frac{2}{p}, \ldots, 1\}.\) Note that, \(\mu_\Delta(\mathbb{K}_t)\) as a function of \(t,\) satisfies the desired property, i.e., \(\mu_\Delta(\mathbb{K}_t)\) is increasing and \(\mu_\Delta(\mathbb{K}_t) < \mu_\Delta(\mathbb{S}_t) = t\) since \(k(k + 1) < pk + \kappa.\)

For the next discussions, we need the assumptions and notations below.

1. Suppose \(\frac{\partial f}{\partial x} \neq 0,\) and \(\{x_t\} = \{x_0\},\)
2. \(\mathbb{T} = \{0\} \cup \mathbb{T} = \{t_j | j = 0, 1, \ldots\},\)
3. \(\alpha(t) = \min_{s \in [x, x]} \|f(s, x)\| , t \in \mathbb{T},\)
4. \(\beta(s) = \max_{s \in \mathbb{S}} \|f(s, x)\| , s \in \mathbb{S},\)
5. \(\gamma(t) = \max_{s \in [x, x]} \|f(s, x)\| , t \in \mathbb{T} \}

**Lemma 3.6.** Let \(f : C \times \mathbb{K} \to \mathbb{R}^N\) be a Carathéodory function i.e., for every \(x \in \mathbb{K}, f(\cdot, x) \in L^1(C),\) and for every \(s \in C, f(s, \cdot) \in C(\mathbb{K}),\) where \(C \subset \mathbb{R}\) is closed and \(\mathbb{K} \subset \mathbb{R}^N\) is compact. Then \(f\) is uniformly bounded on \(C \times \mathbb{K}.\)
Proof. In contrary assume that $f$ is unbounded on $\mathcal{C} \times \mathcal{K}$. Then there must be a sequence \( \{ (c_n, x_n) \} \subset \mathcal{C} \times \mathcal{K} \) with $f(c_n, x_n) \to \infty$. Since $\mathcal{K}$ is compact, there exists an $x \in \mathcal{K}$ for which $x_n \to x$. It implies the existence of an $(n, m) \in \mathbb{N} \times \mathbb{N}$ with the property $1 < n - m$ so that $\| f(c_n, x_n) - f(c_n, x) \| < 1$ and $\| f(c_n, x) \| < m < n < \| f(c_n, x_n) \|$, hence (3.15) since $f(., x)$ is bounded and $f$ is continuous w.r.t $x$. This contradiction proves the assertion. □

**Lemma 3.7.** Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain and $f : \mathbb{S} \times \bar{\Omega} \to \mathbb{R}^N$ is a Carathéodory function. If $0 < \inf_{s \in \mathbb{S}} \mu(s)$, then

(1) $\alpha, \beta \in L^1(\mathbb{S})$,
(2) $\gamma \in L^1(\mathbb{T})$, provided

\[
\gamma(t)\mu(t) \leq \sum_{s \in [t, h(t)]_\mathbb{S}} \beta(s)\mu(s), \quad t \in \mathbb{T}
\]

whereupon we have $\| \gamma \|_{L^1(\mathbb{T})} \leq \| \beta \|_{L^1(\mathbb{S})}$.

Proof.

(1) First we observe that, $f$ is uniformly bounded. For a $\{ x_n \} \subset \bar{\Omega}$ with $x_n \to x$, we have

\[
\lim_{n \to \infty} \sup_{s \in \mathbb{S}} \| f(s, x_n) - f(s, x) \| = 0,
\]

otherwise $\delta \leq \lim_{n \to \infty} \sup_{s \in \mathbb{S}} \| f(s, x_n) \|$ holds for some positive number $\delta$. Taking $\epsilon = \frac{\delta}{4}$, there exists $0 < N$ for which $\frac{3\delta}{4} = \delta - \epsilon \leq \sup_{s \in \mathbb{S}} \| f(s, x_n) - f(s, x) \|$. Also there exists an $s_0 \in \mathbb{S}$ so that

\[
\frac{\delta}{2} \leq \sup_{s \in \mathbb{S}} \| f(s, x_n) - f(s, x) \| - \epsilon \leq \| f(s_0, x_n) - f(s_0, x) \|,
\]

holds for $n > N$ and this is a contradiction. Therefore $\| f(s, x_n) \| \to \| f(s, x) \|$ uniformly. For $x \in \overline{B}(x_0, r)$ let $\psi(x) = \int_{s \in \mathbb{S}} \| f(s, x) \| \Delta s$. We have shown that $\psi(x_n) \to \psi(x)$ and hence $\psi$ attains a maximum and we have

\[
\int_{\mathbb{S}} \max_{x \in B(x_0, r)} \| f(s, x) \| \Delta s = \max_{x \in B(x_0, r)} \psi(x) < \infty.
\]

The integrability of $\alpha$ is clear.

(2) It is immediate from the definition of $\Delta$-integral. □

To satisfy inequality (3.16) it is enough to have

\[
\mu(s) \leq \mu(\sigma(s)), \quad s \in \mathbb{S}
\]

or equivalently if $0 \leq (\sigma - I)^2$ holds on $\mathbb{S}$ where $I$ is the identity function. A special case is $\mathbb{S} = p \mathbb{Z}^+$. 

**Theorem 3.8.** Let $\Omega = B(x_0, r)$ and suppose $\gamma \in L^1(\mathbb{S})$, then the following upper estimation for the frequency of $\mathbb{T}$ in $\mathbb{S}$ holds

\[
\tilde{\mu}_\Delta(\mathbb{T}) \leq \frac{1}{r} \int_{\mathbb{T}} \gamma(t) \Delta t.
\]
Proof. From $r \leq \| f^h(t) \| \gamma(t)$ we obtain
\[
\frac{r}{\gamma(t)} \leq \mu_\Delta([t, h(t)]_\gamma).
\] (3.22)

Note that for $t \in \mathbb{T}$, $\gamma(t) \neq 0$, otherwise $x^*(h(t)) = x_0 + \mu(\rho(h(t)))f(\rho(h(t)), x(\rho(h(t)))) = x_0$, that is a contradiction. The inequality (3.22) implies that
\[
\sum_{j=0}^{\infty} \frac{\mu_\Delta([t_j])}{\mu_\Delta([t, h(t)]_\gamma)} \leq \frac{1}{r} \sum_{j=0}^{\infty} \gamma(t_j) \mu_\Delta([t_j])
= \frac{1}{r} \int_\mathbb{T} \gamma(t) \Delta t.
\] (3.23)

In view of the countable additivity of $\mu_\Delta$ and the inequality $\sum_{j=0}^{n-1} \frac{\mu_\Delta([t_j])}{\sum_{j=0}^{n-1} b_j} \leq \sum_{j=0}^{n-1} \frac{a_j}{b_j}$ ($a_j \in \mathbb{R}^+, b_j \neq 0$, $j = 0, 1, \ldots, n$, $n \in \mathbb{N}$) we have
\[
\frac{\mu_\Delta(\mathbb{T}_\gamma)}{\mu_\Delta(\mathbb{S}_\gamma)} = \frac{\sum_{j=0}^{n-1} \mu_\Delta([t_j])}{\sum_{j=0}^{n-1} \mu_\Delta([t, t_{j+1}])_\gamma}
\leq \frac{\sum_{j=0}^{n-1} \frac{\mu_\Delta([t_j])}{\mu_\Delta([t, t_{j+1}])_\gamma}}{n}, \quad n \in \mathbb{N}
\] (3.24)

After taking the limit, we conclude Eq. (3.21). □

Using a suitable inequality as we did in the proof of Theorem 3.8, we may obtain an upper estimation for the frequency corresponding to an arbitrary bounded domain $\Omega \subset \mathbb{R}^N$. To this end, we need the following lemma.

Lemma 3.9. Suppose $\mu(s) \in [\mu_0, \mu_1]$, $(s \in \mathbb{S})$ with $0 < \mu_0 < \mu_1 < \infty$, $t \in \mathbb{T}$ and $x^*$ is the solution of $\mathcal{P}$. Assume that $f : \mathbb{S} \times \overline{\Omega} \rightarrow \mathbb{R}^N$ is a Carathéodory function, then $\max_{s \in [t, h(t)]_\gamma} \| x^*(s) \| \leq M$ where
\[
M = \sup \left\{ \| u \| \middle| u \in \overline{\Omega} + \mu_1 B(0, R) \right\},
\] (3.25)
\[
R = \max_{t \in \mathbb{T}} |\gamma(t)|.
\] (3.26)

For the next statement, we assume that

1. $\mu(s) \in [\mu_0, \mu_1]$, $s \in \mathbb{S}$
2. $f : \mathbb{S} \times \overline{\Omega} \rightarrow \mathbb{R}^N$ is a Carathéodory function.

Theorem 3.10. Suppose $\gamma \in L^1(\mathbb{S})$. Then the following upper estimation for the frequency of $\mathbb{T}$ in $\mathbb{S}$ holds
\[
\mu_\Delta(\mathbb{T}) \leq \frac{2M}{\omega_0} \int_\mathbb{T} \gamma(t) \Delta t,
\] (3.27)
where $\omega_0 = \min_{s \in \mathbb{S}} \| u \|^2 - \| x_0 \|^2$. 

Proof. Using integration by parts on time scales for a \( t \in \mathbb{T} \) we obtain
\[
\left| \int_{t_i}^{t_f} \left( x^\sigma(s) + x^{\omega\sigma}(s) \right) x^{\omega\Delta}(s) \Delta s \right| = \left| \left\| x^\sigma(s) \right\|^2 \right|_{h(t_i)}^{h(t_f)} \leq \left| \left\| x^\sigma(h(t)) \right\|^2 - \left\| x_0 \right\|^2 \right|, 
\]
where \( x^\sigma \) is the solution of \( \mathcal{P} \). It implies that
\[
\omega_0 \leq \left| \int_{t_i}^{t_f} \left( x^\sigma(s) + x^{\omega\sigma}(s) \right) x^{\omega\Delta}(s) \Delta s \right|, \quad t \in \mathbb{T}. 
\]
From the upper estimation
\[
\left| \int_{t_i}^{t_f} \left( x^\sigma(s) + x^{\omega\sigma}(s) \right) x^{\omega\Delta}(s) \Delta s \right| \leq 2M \mu_\Delta (|t|, h(t)) \gamma(t), 
\]
and Eq. (3.29) we obtain
\[
\frac{\mu_\Delta (|t|)}{\mu_\Delta (|t|, h(t))} \leq \frac{2M \gamma(t)}{\omega_0}. 
\]
Using the same deduction as in the proof of Theorem 3.8 gives the result. □

The Opial-type inequality on time scales is discussed in [7]. This inequality provides a shortened estimation which amends the upper bound in Theorem 3.8 in special case \( \epsilon < 1 \) and \( x_0 = 0 \).

**Theorem 3.11.** Suppose \( \gamma \in L^1(\mathbb{T}) \) and \( x_0 = 0 \). Then an upper estimation for the frequency of \( \mathbb{T} \) in \( \mathbb{S} \) is
\[
\tilde{\mu}_\Delta (\mathbb{T}) \leq \frac{1}{\sqrt{\omega_0}} \int_{\mathbb{T}} \gamma(t) \Delta t. 
\]

**Proof.** Similar to the proof of the Opial-type inequality I (Theorem 3.1.1, [2]), let \( w(s) = \int_{t_i}^{t_f} \left\| x^{\omega\Delta}(\tau) \right\| \Delta \tau, \) then \( w_\Delta(s) = \left\| x^{\omega\Delta}(s) \right\| \) and \( \left\| x^\sigma(s) \right\| \leq w(s) \). Also we have
\[
\left| \int_{t_i}^{t_f} \left( x^\sigma(s) + x^{\omega\sigma}(s) \right) x^{\omega\Delta}(s) \Delta s \right| \leq \int_{t_i}^{t_f} \left( \left\| x^\sigma(s) \right\| + \left\| x^{\omega\sigma}(s) \right\| \right) \left\| x^{\omega\Delta}(s) \right\| \Delta s 
\]
\[
\leq \int_{t_i}^{t_f} \left( w(s) + w^{\omega\sigma}(s) \right) w^{\omega\Delta}(s) \Delta s 
\]
\[
= \int_{t_i}^{t_f} \left( w^2(s) \right)^{\Delta} \Delta s = w^2(h(t)) 
\]
\[
= \left( \int_{t_i}^{t_f} \left\| x^{\omega\Delta}(s) \right\| \Delta s \right)^2 
\]
\[
\leq \mu_\Delta^2 (|t|, h(t)) \gamma^2(t), 
\]
where in the 4th line we have used the Cauchy–Schwarz inequality. Using Eq. (3.29) we obtain
\[
\sqrt{\omega_0} \gamma(t) \leq \mu_\Delta (|t|, h(t)) \gamma(t). 
\]
\[
\text{The rest of the proof is similar to the proof of Theorem 3.8.} \quad \square
\]
Example 3.12. Suppose $\sigma(s) \leq s + \frac{1}{s}$ holds for $s \in \mathbb{S}$ and let $f_i(s, x) = \frac{x_i^2}{1+x^2}$, $i = 1, 2$. Then we have $\beta(s) = \max_{x \in B(s, r)} \frac{\sqrt{x_1^4 + x_2^4}}{1+x^2} \leq \frac{r^2}{1+r^2}$, and $\gamma(t) \leq \frac{r^2}{1+r^2}$. From Theorem 3.8, we obtain

$$\tilde{\mu}_\Delta (\tilde{T}) = \frac{1}{r} \int_{\tilde{T}} \gamma(t) \Delta t$$

$$\leq r \int_{\tilde{T}} \Delta t \frac{1}{1+t^2}$$

$$\leq r \left( \int_{[0,t_1)} \Delta t + \int_{[t_1, \infty)} \frac{\Delta t}{t \sigma(t)} \right)$$

$$= r \left( t_1 - \int_{[t_1, \infty)} \frac{1}{t} \Delta t \right)$$

$$= r \left( t_1 + \frac{1}{t_1} \right).$$

(3.34)

Example 3.13. Let $\mathbb{S} = \mathbb{Z}^+$, $f(s, x) = (\sin(0.2 + x_2), \cos(0.2 + x_1))$, $x_0 = (0, 0)$ and $\Omega = B(0, 2)$. Since the system is autonomous, the controlled system is periodic. The first control time is $t_1 = 4$ and the frequency is $\tilde{\mu}_\Delta (\tilde{T}) = 0.25$. Fig. 2 illustrates $x^*$ and $x$.

3.1. On a uniform distribution mod 1

Let $\pi = \{a_t\} \subset [0, 1]$ where $a_t = \frac{\mu_\Delta (\tilde{T})}{\mu_\Delta (\tilde{S}_t)}$ and $A_t$ denote the equivalent class of $a_t$ defined by

$$A_t = \left\{ \tau \in \mathbb{T} \left| \frac{\mu_\Delta (\tilde{T})}{\mu_\Delta (\tilde{S}_t)} = a_t \right\} \right.$$

(3.35)
For a $u \in \pi$, let $t(u) = \min A_u$.

Suppose that $\pi$ is uniformly distributed in $[a, b]_{\pi} \subset [0, 1]$ for some $a, b \in \pi$ with $a < b$ and we express it shortly by u.d in $[a, b]_{\pi}$. The aim of this subsection is to obtain a necessary condition for u.d of $\pi$ in terms of a critical dynamic of the controlled system’s solution $x$.

**Definition 3.14.** Let $\mathbb{U} \subset [0, 1]$ be a time scale. A sequence $\{u_n\} \subset \mathbb{U}$ is said to be distributed uniformly in $[0, 1]_{\mathbb{U}}$ if for every $a, b \in \mathbb{U}$ with $a < b$

$$\lim_{N \to \infty} \frac{\sharp \{1 \leq n \leq N \mid u_n \in [a, b]\}}{N} = b - a,$$

(3.36)

where $\sharp$ denotes the cardinality.

For a notational simplicity, the sequence $\pi$ is denoted by $\{u_n\}$. The following elementary lemma is verified easily.

**Lemma 3.15.** Suppose $\varphi : \mathbb{T} \times [0, d] \to \mathbb{R}^+$ with $d = \sup_{x \in \mathbb{P}} \|x\|$, is a function with the following properties

1. for every $t \in \mathbb{T}$, $\varphi(t, \cdot)$ is continuous,
2. $\sum_{t \in \mathbb{T}} \varphi(t, \cdot)$ is convergent uniformly on $[0, d]$.

Then $\varphi$ is uniformly bounded and for any sequence $\{r_n\} \subset [0, d]$, we have $\varphi(t_n, r_n) \to 0$ as $n \to \infty$.

**Proposition 3.16.** The function $\theta : [a, b]_{\pi} \to \mathbb{R}^+$ defined by

$$\theta(u) = \begin{cases} \varphi(t(u), \|x^\theta(t(u))\|), & u \in \pi \\ 0, & u \not\in \pi \end{cases}$$

(3.37)

is $\Delta$-integrable in the Riemann sense.

**Proof.** Since $\theta$ is bounded on $[a, b]_{\pi}$, it is enough to show that for a given positive $\epsilon$ there exists a positive $\delta$ so that for any partition of $[a, b]_{\pi}$ denoted by $P$ with $P \in P_\delta$, $U(\theta, P) - L(\theta, P) \leq \epsilon$ holds (Theorem 5.9, [8]). Let

$$\delta = \epsilon \sum_{t \in \mathbb{T}} \varphi(t, r_t),$$

(3.38)

where $\varphi(t, r_t) = \max_{x \in \mathbb{P}} \varphi(t, \|x\|)$. Let $I$ denotes the index set corresponding to those subintervals which satisfy $u_{i+1} - u_i \leq \delta$ and $J$ denotes the index set corresponding to those subintervals satisfying $u_{i+1} - u_i < \delta$ and $\rho_{\pi}(u_{i+1}) = u_i$, where $\rho_{\pi}$ is the step backward function of the time scale $\pi$. Let

$$M_i = \max_{u \in [u_i, u_{i+1}]_{\pi}} \theta(u), \quad m_i = \min_{u \in [u_i, u_{i+1}]_{\pi}} \theta(u).$$

(3.39)

Then for any $P \in P_\delta$ we have

$$U(\theta, P) - L(\theta, P) = \sum_{i \in I} (M_i - m_i)(u_{i+1} - u_i) + \sum_{i \in J} (M_i - m_i)(u_{i+1} - u_i)$$

$$= \sum_{i \in I} (M_i - m_i)(u_{i+1} - u_i).$$
\[
\begin{align*}
\leq & \delta \sum_{i \in l} (M_i - m_i) \\
\leq & \delta \sum_{i \in l} M_i \\
\leq & \delta \sum_{i \in \mathbb{T}} \varphi(t, \tau) = \epsilon.
\end{align*}
\]

This completes the proof. \(\square\)

Note that Lemma 3.15 does not imply continuity of \(\theta\) since \(\pi\) may intersect its limit set. The following theorem is a time scale counterpart of a classical theorem on uniform distribution (see e.g., Corollary 1.1, [15]). The proof is similar to the classical case and it is omitted.

**Theorem 3.17.** The sequence \(\pi\) is u.d in \([a, b]_{\pi}\) if and only if

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \theta(u_n) = \int_{a}^{b} \theta(u) \Delta u,
\]  
(3.40)

holds for every \(\Delta\)-integrable function \(\theta : [a, b]_{\pi} \to \mathbb{R}^+\).

In view of Lemma 3.15 and Proposition 3.16, for the specific function \(\theta\) defined by Eq. (3.37), the statement below is verified.

**Theorem 3.18.** Suppose \(\pi\) is u.d in \([a, b]_{\pi}\). Then for the critical dynamic of the controlled system, i.e., \(\{x^\theta(t)\}\), we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \varphi(t(u_n), \|x^\theta(t(u_n))\|) = \int_{a}^{b} \varphi(u, \|x^\theta(t(u))\|) \Delta u.
\]  
(3.41)

4. Conclusion

In this work, with a discrete Dirac Delta function corresponding to a connected bounded domain, an impulsive feedback control on an isolated time scale is considered. A shift-invariance property of the graininess function of the given shift-invariant time scale is deduced and periodicity issues in the special case \(\Omega = B(x_0, r)\) are discussed.

Considering the control times as a time scale at which the non-controlled system’s solution jumps out of \(\Omega\), an occurrence frequency of this sequence of times in the main time scale is introduced. Then with the help of dynamic inequalities on time scales, upper estimations for the mentioned frequency are obtained.

A uniformity of distribution mod 1 of the control times is considered. Then via a time scale counterpart of a classical theorem on u.d mod 1 and the Riemann \(\Delta\)-integrability of \(\theta\) defined by Eq. (3.37), an identity on the critical states of the controlled system is derived.

**References**


