Recursive dynamics in a class of systems on q-timescales with unbounded delays

Mehdi Nategh

Department of Mathematics & Statistics, Missouri S&T, Rolla, MO 65401, US

A R T I C L E   I N F O

Article history:
Received 29 January 2018
Received in revised form 16 October 2018

Keywords:
Time scales
Retarded equations
Unbounded delay
Recursive pattern

A B S T R A C T

This work deals with a class of retarded equations on q-time scales which involves unbounded delays. A global exponential stability result is obtained and a dynamic of the recursive steps due to the delay function is discussed. A formal representation of natural numbers with respect to an unbounded delay function is featured with an asymptotic behavior of the recursion in an extended class of retarded equations.

© 2018 Elsevier B.V. All rights reserved.

1. Introduction

This paper presents a discussion on a recursive dynamic governed by a class of retarded equations of the form

$$u(k) = q^\alpha \lambda(k) \sum_{n=0}^{\lfloor \alpha k \rfloor} q^{\alpha n} u(n) K(k, n),$$

(1.1)

where $\alpha \in (0, 1)$, $q \in [0, \frac{\sqrt{5} - 1}{2}]$, $q^* = \frac{q^{\alpha} - 1}{q}$ and $\lambda$ and $K$ are to be known later. Eq. (1.1) is reminiscent of the trigonometric series

$$f(t) = \sum_{n=1}^{\infty} b^{-n\alpha} \cos(b^n t), \quad 0 < \alpha < 1, \ b > 0$$

(1.2)

which is the no-where differentiable (Weierstrass) function [1].

Systems of equations with single or multiple time lag inevitably emerge from the modeling in a diverse area from ecosystems, biosystems and physiology to a variety of control systems as well [2–4]. It is evident that the time has a discrete nature in the real world applications. Also, the fact that time is as important as the state in a variety of systems, supports deliberations on non-autonomous difference equations and timescales counterparts of the mentioned class of retarded systems [5–7].

An example of a multiple delayed system is introduced as [8]

$$x(t) = \sum_{j=1}^{m} A_j x(t - h_j) + \int_{-\tau}^{0} G(\theta) x(t + \theta) d\theta,$$

(1.3)
where $A \in \mathbb{R}^{N \times N}$, $j = 1, 2, \ldots, m$, $0 < h_1 < h_2 < \cdots < h_m$, $\tau \in \mathbb{R}^+$ and $G : [\tau, 0] \to \mathbb{R}^{N \times N}$ is piecewise continuous. Replacing the real line with an appropriate isolated timescale $\mathbb{T}$, a discretized Eq. (1.3) would be

$$x(t) = \sum_{j=1}^{m} A_j x(t - h_j) + \sum_{\theta = -\tau}^{\mu(0)} G(\theta) x(t + \theta) \mu(\theta), \quad t, -\tau, \theta \in \mathbb{T}$$

(1.4)

In this sequel a more extended form of Eq. (1.4) which is a non-autonomous equation of retarded type with an unbounded time lag is considered as follows

$$u(k) = \lambda(k) F \left( (0, 1, \ldots, f(k)), \{u(j)\} | j = 0, 1, \ldots, f(k) \right),$$

(1.5)

where $\lambda : N_0 \to N_0$ and $F : \mathcal{P}(N_0) \times X \to \mathbb{R}^N$ are functions, $\mathcal{P}(.)$ is the power set,

$$X = \left\{ u : N_0 \to \mathbb{R}^N \mid \|u(n)\| \leq C e_{\mathcal{O}r}(n, 0) \text{ for some } C \in \mathbb{R}^+ \right\},$$

(1.6)

for $r \in \mathbb{R}^+$, $e_{\mathcal{O}r}(n, 0)$ is the timescale exponential function and $f : N_0 \to N_0$ is an unbounded non-monotone and onto function satisfying $f(k) < k$ ($k \in N$). Eq. (1.1) is a special case of Eq. (1.5) which is defined on a $q$-time scale $\mathbb{T}$ and will be studied in Section 2. See [9] for a quick overview on dynamics on timescales and [10] for a complete reference on timescales calculus.

Since the main results provided in Section 3 address the recursive dynamic of a stable solution, we need to ensure the existence of such solution. In Theorem 2.3, a closed form solution of Eq. (1.1) is obtained and it led to a global exponential stability result. This provision is adequate for the main results discussed in Section 3.

The statements of Theorems 3.1–3.3 formulate the main result of the present work. In the retarded equation (1.1), the $k$th state $u(k)$ is a function of the first $\lfloor ak \rfloor$ states. It makes us to consider the dynamic of this backward dependence, that is to consider the dynamic of the ratio $\frac{h}{\tau}$ and in the extended system (1.5), the ratio $\frac{f(k)}{\tau}$ where $f$ is supposed to be the delay function.

2. Global exponential stability

The set of positive integers including zero is denoted by $N_0$, $\mathbb{R}^+$ denotes the positive real numbers $(0, \infty)$ and $\mathbb{Q}$ and $\mathbb{Q}^c$ stand for rational and irrational numbers respectively.

By a $q$-time scale $\mathbb{T}$ we mean a closed subset of the real line introduced by $\mathbb{T} = \{0\} \cup \bigcup_{n \in N_0} q^n \mathbb{N}_0$, for some $q \in \mathbb{R}^+$ and a non-constant function $j : N_0 \to N_0$. The calculus on $q$-timescales is an example of a calculus without limit. Another example is the $h$-difference calculus and these two examples can be related by $q = e^h$ when $q$ is considered as the first letter of "quantum" and $h$ is the Plank’s constant. As an application, a description of even–even nuclei’s spectra using a quantum counterpart of special unitary groups denoted by $SU_q(2)$ is reported. For a detailed report on $q$-calculus in physics see [12, [11].

In order to start off the discussion we need the following notations (Fig. 1.)

$$\mathbb{T}_1 = q^{N_0}, \quad \mathbb{T}_1^0 = q^{N_0+1}, \quad \mathbb{T} = (0) \cup \mathbb{T}_1 \cup \bigcup_{n \in N_0} (q^n + \mathbb{T}_1),$$

(2.1)

Consider an eigenvalue problem defined on $\mathbb{T}_1 \subset \mathbb{T}$

$$x(t) = \lambda \int_{\mathbb{T}_1 \cup \{0\}} x \left( \frac{t}{t \mid q \tau} \right) dq \tau, \quad t \in \mathbb{T}_1$$

$$x(1) = x_1.$$  

(2.2)

The finite terms summation of the primal $q$-integral equation appears as the delay difference equation (2.2) with unbounded delay $\rho(k) = \{ak\} + (1 - \alpha)k$, where $\lfloor ak \rfloor$ is the fractional part of $\alpha k$ and we obtain

$$x(q^{\lfloor ak \rfloor}) = \lambda q^n \sum_{n=0}^{\lfloor ak \rfloor} q^n x(q^{ak-n}).$$

(2.3)
See [1] for local fractional derivative of a non-differentiable functions on $\mathbb{T}$ and their equivalent finite sum representations. See [11,12] for theory of integration and differentiation on q-time scales.

For $k$ satisfying $\lfloor ak \rfloor = 0$, we have $x_1 = \lambda x_1$, that is $\lambda q^* = 1$. After removing $x_1$ and utilizing a notation $u(k, n) = x(q^k-n)$ for $k$ satisfying $\lfloor ak \rfloor > 0$ and $n \in \mathbb{N}_0$, Eq. (2.3) implies

$$\sum_{n=1}^{\lfloor ak \rfloor} q^n u(k, n) = 0. \quad (2.4)$$

Problem (2.2) has infinitely many solutions. Indeed for a real $u_0$, all vectors $(u_0, q^{\lfloor (ak-1) \rfloor}, 0, \ldots, 0, -1) \in \mathbb{R}^{\lfloor ak \rfloor + 2}$ satisfy (2.3). However in this trivial case we have the following statement that describes a universal property of vanishing summations.

Theorem 2.1. Suppose Eq. (2.4) holds and let

$$a_k = \max_{n \in \{1, \ldots, \lfloor ak \rfloor \}} u(k, n), \quad b_k = \min_{n \in \{1, \ldots, \lfloor ak \rfloor \}} u(k, n), \quad (2.5)$$

then

$$|b_k| \leq \frac{\lfloor ak \rfloor}{q^{\lfloor ak \rfloor}} a_k. \quad (2.6)$$

holds for those $k \in \mathbb{N}$ satisfying $0 < \lfloor ak \rfloor$.

Proof. Assume that $u$ is not equivalent to zero. Since the summation (2.4) consists of positive and negative terms and $q < 1$, then we have

$$0 = \sum_{n=1}^{\lfloor ak \rfloor} q^n u(k, n) \leq \sum_{u(k,j) \geq 0}^{\lfloor ak \rfloor} u(k, n) + \sum_{u(k,j) < 0}^{\lfloor ak \rfloor} q^{\lfloor ak \rfloor} u(k, n)$$

$$\leq \lfloor ak \rfloor a_k + b_k q^{\lfloor ak \rfloor}, \quad (2.7)$$

that is, $-b_k = |b_k| \leq \frac{\lfloor ak \rfloor}{q^{\lfloor ak \rfloor}} a_k$. $\square$

For $X$ as a Banach algebra, we consider the main boundary value problem of this manuscript involving a singular q-integral equation of the form below

$$x(t) = \lambda(t) \int_{t \cup \{0\}} x(\tau) K(\frac{t}{\tau^{1/\alpha}}) d_q \tau, \quad t \in \mathbb{T}$$

$$x(1) \neq 0, \quad \lim_{t \to 0^+} x(t) = 0, \quad (2.8)$$

where $K : \mathbb{R} \to X$ is the kernel that vanishes outside of $[0, 1], K(t) \neq 0$ for $t \in \mathbb{T}$, and $\lambda : \mathbb{T} \to X$ is a function. It is assumed that, $K$ is bounded on $[0, 1]$ with $K_1 = \sup_{t \in [0,1]} |K(t)|$ and $K_0 = \inf_{t \in [0,1]} |K(t)| \neq 0$.

First we consider problem (2.8) on $\mathbb{T}_1$. Making use of the notations $u(k) = x(q^k), \lambda(k) = \lambda(q^k), K(k, n) = K(q^n-k)$, we obtain an equivalent representation of Eq. (2.8)

$$u(k) = q^* \lambda(k) \sum_{n=0}^{\lfloor ak \rfloor} q^n u(n) K(k, n), \quad (2.9)$$

that is a recursive equation.

The boundary values in Eq. (2.8) take the form of $u(0) \neq 0$ and $\lim_{k \to +\infty} u(k) = 0$. The boundary condition $u(0) \neq 0$ is essential otherwise, $u(0) = 0$, gives the zero solution.

An objective in the study of Eq. (2.9) is stability of a solution. We denote the exponential function on $\mathbb{N}_0$ by $e_p(t, s)$ where $p : \mathbb{N}_0 \to \mathbb{R}$ is regressive and $t, s \in \mathbb{N}_0$ (see §2, [13]).

Definition 2.1. For a given $u_0 = u(0) \in X$, the function $u : \mathbb{N}_0 \to X$ satisfying Eq. (2.9), is simply called a solution. The zero solution of Eq. (2.9) is called globally exponentially stable solution if there exist positive constants $\gamma, C$, so that

$$|u(n)| \leq Ce^{\gamma p(n, 0)}, \quad n \in \mathbb{N}_0 \quad (2.10)$$

for any $u(0) \in \mathbb{R}$.

Based on assuming the existence of a non-trivial solution, in the special case $X = \mathbb{R}$, a similar inequality as (2.6) holds in an asymptotic sense below
Theorem 2.2. Suppose \( X = \mathbb{R}, \lambda \equiv 1/q^* \) and \( u \) is a non-trivial solution of Eq. (2.9) which satisfies \( \lim_{k \to +\infty} u(k) = 0 \). Let
\[
a_k = \max_{n \in \{0, 1, \ldots, |\alpha k|\}} u(n), \quad b_k = \min_{n \in \{0, 1, \ldots, |\alpha k|\}} u(n).
\]
then for \( k \) large enough, the following inequality holds
\[
b_k \leq \frac{1}{K_0 q^{|\alpha k|}} \left\{ K_1 |\alpha k| a_k + 1 \right\}.
\]
Proof. Let \( \omega(k, n) = u(n)K(k, n) \) with \( \tilde{a}_k = \max_{n \in \{0, 1, \ldots, |\alpha k|\}} \omega(k, n) \) and \( \tilde{b}_k = \min_{n \in \{0, 1, \ldots, |\alpha k|\}} \omega(k, n) \). Using the limit condition, for a given \( \epsilon > 0 \) there exists a large enough lower bound \( L \) for which, \( \sum_{n=0}^{|\alpha k|} q^{|\alpha u(n)}K(k, n)| \leq \epsilon \), holds if \( L < k \). There must be both positive and negative \( \omega(k, n) \) since \( u \) is non-trivial. We have the following upper and lower estimations
\[
b_k \leq -\tilde{b}_k, \quad \tilde{a}_k \leq a_k K_1,
\]
since \( \tilde{b}_k \) should be negative. Similar to the proof of Theorem 2.1, we obtain
\[
-\epsilon < \sum_{n=0}^{|\alpha k|} q^{|\alpha u(n)}K(k, n) \leq |\alpha k| \tilde{a}_k + \tilde{b}_k q^{|\alpha k|}.
\]
Using (2.13) in (2.14), gives the result. \( \Box \)

404

Let \( \gamma > 1 \) and suppose that
\[
X_{\gamma} = \left\{ u : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R} \left| u(k, n) \leq Ce_{\gamma}(2k, n) \text{ for some } C \in \mathbb{R}^+ \right. \right\}.
\]
Note that \( e_{\gamma}(n, 0) = \frac{1}{(1 + \gamma)^n} \), therefore \( \lim_{n \to +\infty} |u(n)| = 0 \), for \( u \in X \). The linear space \( X_{\gamma} \) is a Banach space.

Theorem 2.3. Suppose \( \lambda(k)K(k, n) \in X_{\gamma} \), then the zero solution of Eq. (2.9) is globally exponentially stable.

Proof. Let \( \beta(k, n) = q^* \lambda(k)K(k, n) \). For \( u_0 \in X \), it can be observed that, \( u(k) = Q_k(q^*)u_0 \), where \( Q_k \in \mathbb{R}[q^*] \), i.e.,
\[
u(k) = u_0 \sum_{n=0}^{q(k)} q^{\alpha n} p(k, n),
\]
and the coefficients \( p(k, n) \) consist of products of the form \( \prod_{i<j} \beta(j, i) \) and \( q(k) = \deg(Q_k) \). The degree of \( Q_k \) satisfies the difference equation \( q(k) = k - 1 + q(k - 1) \), and hence \( q(k) = \frac{k(k-1)}{2} \). For example if \( |4\alpha| = 3 \), then
\[
\begin{align*}
p(4, 6) &= \beta(4, 3)\beta(3, 2)\beta(2, 1)\beta(1, 0), \\
p(4, 5) &= \beta(4, 3)\beta(3, 2)\beta(2, 0), \\
p(4, 4) &= \beta(4, 3)\beta(3, 1)\beta(1, 0), \\
p(4, 3) &= \beta(4, 2)\beta(2, 1)\beta(1, 0) + \beta(4, 3)\beta(3, 0), \\
p(4, 2) &= \beta(4, 2)\beta(2, 0), \\
p(4, 1) &= \beta(4, 1)\beta(1, 0), \\
p(4, 0) &= \beta(4, 0).
\end{align*}
\]

The total number of products \( \prod_{i<j} \beta(j, i) \) is \( 2^{k-1} \) provided \( |\alpha k| = k - 1 \). This is the number of paths in a complete graph of degree \( |V| = k + 1 \) joining two vertices labeled \( k \) and \( 0 \) and having the property that, the sequence of labels in each path is arranged in a descending order. Since we have \( \frac{k-1}{2} + 1 \) terms in the polynomial \( Q_k \), then the maximum number of products \( \prod_{i<j} \beta(j, i) \) appearing in the coefficient of a \( q^{\alpha n} \) \( (0 \leq n \leq q(k)) \), is \( 2^{k-1} - \frac{k(k-1)}{2} \). From the assumption on \( \beta \) and the definition of \( X_{\gamma} \) we infer that
\[
|p(k, n)| \leq \left( 2^{k-1} - \frac{k(k-1)}{2} \right) \max_{0 \leq m \leq |\alpha k|} |\beta(k, m)|
\]
\[
\leq C_0 \left( 2^{k-1} - \frac{k(k-1)}{2} \right)e_{\gamma}(k, 0), \quad n = 0, 1, \ldots, \frac{k(k-1)}{2}.
\]
This completes the proof. \( \Box \)

Remark 2.1. It is easy to verify that, with the same deduction as Theorem 2.3 and with the help of an appropriate notation, for every \( n \in \mathbb{N} \), problem (2.8) defined on \( \{0\} \cup \left(q^n + \frac{\gamma}{\gamma'}\right) \) possesses a globally exponentially stable solution. Therefore, we conclude that, the solution of problem (2.8) defined on \( \mathbb{T} \) is globally exponentially stable.
Remark 2.2. Theorem 2.3 shows that, $\beta$ is solely responsible for the stability. Hence, modifying the assumption on $\beta$ may give rise to different stability modes. For example, a bounded $\beta$, results in a Lagrange stability. It is also understood from the closed form solution (2.16) that, any stability result obtained from a certain class of $\beta$ will hold globally.

3. Dynamic of the recursion

It is well-known that, for an irrational number $\alpha \in (0, 1)$, the sequence of fractional parts of $\alpha n$, i.e., $\{\alpha n\} = \alpha n - \lfloor \alpha n \rfloor \ (n \in \mathbb{N})$ is distributed uniformly in $(0, 1)$ in the sense that [14]

$$
\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}\{n \mid 1 \leq n \leq N, \alpha n - \lfloor \alpha n \rfloor \in [a, b)\} = b - a,
$$

where $\mathbb{1}$ denotes the cardinality.

**Definition 3.1.** For the function $f_\alpha : \mathbb{N} \to \mathbb{N}_0$, defined by $k \mapsto \lfloor \alpha k \rfloor$, let the jump function $v_\alpha : \mathbb{N} \to \mathbb{N}$ be defined by

$$
v_\alpha(n) = \lfloor \beta f_\alpha^{-1}(n - 1) \rfloor = \lfloor i \in \mathbb{N}_0 \mid \alpha i = n - 1 \rfloor.
$$

The function $v_\alpha$ is bounded since for an arbitrary $i, n, k \in \mathbb{N}_0$ which satisfy $\lfloor \alpha i \rfloor = \lfloor \alpha(i + N) \rfloor = k$, we have $\alpha i = k + \delta_0$ and $\alpha(i + N) = k + \delta_i, \delta_0, \delta_i \in [0, 1)$, and we obtain $N = \frac{\delta_i - \delta_0}{\alpha}$. Let us denote the possible values of $v_\alpha$ by

$$
v_\alpha(N) = \{l_1, l_2, \ldots, l_T\},
$$

for some $T = T(\alpha) \in \mathbb{N}$ where $l_i = \min\{m \in \mathbb{N} \mid \lfloor \alpha m \rfloor = i\}, i = 1, 2, \ldots, T$, and $l's$ are not necessarily distinct.

**Example 1.** 1. $v_{0.4}(N) = \{2, 3\}$ and it is a 2-periodic function which takes values $3, 2, 3, 2, \ldots$. Note that, $T(0.4) = 2$ and $\frac{1 + 0}{0.4} = \frac{1}{0.4}$.

2. $v_{6/13}(N) = \{2, 3\}$ and $T(6/13) = 6$ with $l_1 = 3, l_2 = 2$ for $i = 2, \ldots$, 6. Also $\frac{1}{(6/13)} \sum_{i=1}^{6} l_i = \frac{13}{6}$.

**Lemma 3.1.** Suppose $v_\alpha(N) = \{l_i\}_{i=1}^{T(\alpha)}$, then there exists at least one index $i \leq T(\alpha)$ for which $1 < l_i$.

**Proof.** Based on the Dirichlet approximation theorem, for a given $\alpha \in \mathbb{Q}^c$, there exist infinitely many $n, m \in \mathbb{N}$, which satisfy $|\alpha - \frac{n}{m}| < \frac{1}{m^2}$ (Theorem 5.1.1 and Exercise 5.1.2, [15]). One obtains $\lfloor \alpha m \rfloor = n$, since $n \leq \alpha m < n + \frac{1}{m}$. Moreover $\alpha(m + 1) - n < \frac{n}{m} + \frac{1}{m} + \frac{1}{m^2} < 1$, provided $m$ is large enough. That is $\lfloor \alpha m \rfloor = \lfloor \alpha(m + 1) \rfloor = n$, and we infer that $1 < \max_{n \in \mathbb{N}} v_\alpha(n)$. □

**Lemma 3.2.**

1. Suppose $\alpha \in \mathbb{Q}$, then $v_\alpha$ is constant if and only if $\alpha = \frac{1}{p}$ for $p \in \text{Primes}$. In this case, we have $v_{1/p}(N) = \{p\}$.

2. $v_\alpha$ is periodic if and only if $\alpha \in \mathbb{Q}$.

**Proof.** The first argument is immediate by the definition of $v_\alpha$. For the second, it can be verified that, $v_\alpha$ is periodic for rational $\alpha$. Now assume that $v_\alpha$ is periodic, i.e., $v(n) \in \{l_{i+1 \mod T}\}$ for $T \in \mathbb{N}$ and $\alpha \in \mathbb{Q}^c$.

It is easy to verify that, every natural number $m$ has a representation of the form

$$
\begin{align*}
m &= m_0 + \sum_{i=1}^{j} l_i + n(m) \sum_{i=1}^{T} l_i \\
m_0 &= 0, 1, \ldots, l_{i+1} - 1, \quad j = 0, 1, \ldots, T - 1; \quad n(m) = 0, 1, 2, \ldots. \\
n(m) &\to +\infty, \quad \text{as } m \to +\infty
\end{align*}
$$

with the convention that $l_0 = 0$. It can be obtained easily by writing the numbers $\mathbb{N}_0$ in accordance with the sequence $\{v_\alpha(n)\}$. This representation gives rise to

$$
\begin{align*}
\alpha \left( m_0 + \sum_{i=1}^{j} l_i + n(m) \sum_{i=1}^{T} l_i \right) &= \alpha \left( \sum_{i=1}^{j} l_i + n(m) \sum_{i=1}^{T} l_i \right) \\
&= n(m)T + j.
\end{align*}
$$

Let us observe that

$$
\begin{align*}
\lfloor \alpha m \rfloor &= \alpha m - \lfloor \alpha m \rfloor = \alpha m_0 + \alpha (m - m_0) - \lfloor \alpha m \rfloor \\
&= \alpha m_0 + \alpha (m - m_0) - \lfloor \alpha (m - m_0) \rfloor \\
&= \alpha m_0 + \lfloor \alpha (m - m_0) \rfloor.
\end{align*}
$$
We infer that \( \{\alpha(m - m_0)\} \leq \{\alpha m\} \). The sequence \( \{\alpha m\} - \{\alpha(m - m_0)\} \) is nonzero since by Lemma 3.1, \( 0 < m_0 \) for some \( m \in \mathbb{N} \). We conclude that, \( \{\alpha m\} \) is an increasing sequence and therefore, it has only one limit point in \( (0, 1) \). This contradicts with the uniform distribution of \( \{\alpha m\} \) in \( (0, 1) \) while \( \alpha \) is assumed to be irrational. The assertion now is proved. \( \square \)

According to the representation (3.4), we have the following statement:

Theorem 3.1. Suppose \( \alpha \in \mathbb{Q} \) then \( \frac{1}{T(\alpha)} \sum_{i=1}^{T(\alpha)} l_i = \frac{1}{\alpha} \). 

Proof. In view of the representation mentioned in the proof of Lemma 3.2(2), that is \( k = k_0 + \sum_{j=1}^{T} l_i + n(k) \sum_{i=1}^{T} l_i \), where \( n(k) \to +\infty \) as \( k \to +\infty \), we obtain

\[
\lim_{k \to +\infty} \frac{k}{T(\alpha)} = \lim_{k \to +\infty} \frac{k_0 + \sum_{j=1}^{T} l_i + n(k) \sum_{i=1}^{T} l_i}{j + n(k)T(\alpha)} = \frac{1}{T(\alpha)}. 
\]

We utilize

\[
\lim_{k \to +\infty} \frac{|\alpha k|}{k} = \lim_{k \to +\infty} \frac{\alpha k - \{\alpha k\}}{k} = \alpha.
\]

and this completes the proof. \( \square \)

This is the periodicity of \( v_\alpha \) with rational \( \alpha \), that provides the representation (3.4). Then the sequence of ratios \( \frac{|\alpha k|}{k} \) is finite up to the equivalence classes defined by \( \{\alpha k\} \in \mathbb{Z}_{T(\alpha)} \) and \( k \in \mathbb{Z} \).

In case \( \alpha \in \mathbb{Q}^c \), still we can conclude a similar result. Let us denote the \( v_\alpha(\mathbb{N}) \) with irrational \( \alpha \) by a sequence \( (l_i)_{i\in\mathbb{N}} \). Although the \( v_\alpha \) is aperiodic, the boundedness provides exists of a finite \( T(\alpha) \) defined by

\[
T(\alpha) = \min \left\{ i \in \mathbb{N}_0 \mid l_i \in \{l_1, l_2, \ldots, l_T\}, \ j \in \mathbb{N}_{T+1} \right\}. \tag{3.9}
\]

It is clear from Lemma 3.2(1) that, for \( \{l_1, l_2, \ldots, l_T\} \), there exists \( 1 \leq i, j \leq T(\alpha) \) so that \( i \neq j \).

Let \( P_{\nu_\alpha} : \mathbb{N} \times \mathbb{N} \to \{0, 1\} \) denote a function defined by

\[
P_{\nu_\alpha}(i, j) = \begin{cases} 1, & \text{ith entry of the sequence } \{l_i\} \text{ is } l_j \\ 0, & \text{otherwise} \end{cases} \tag{3.10}
\]

Obviously \( i, j = 1, 2, \ldots, T(\alpha) \)

\[
\begin{align*}
\left\{ P_{\nu_\alpha}(j, j) = 1, \\
\sum_{j=1}^{T(\alpha)} P_{\nu_\alpha}(i, j) = \mathbb{Z}\{l_j | l_j = l_i, \ j = 1, 2, \ldots, T(\alpha)\}. \right. \tag{3.11}
\end{align*}
\]

To obtain a representation for \( k \in \mathbb{N} \), we sort the numbers \( \mathbb{N}_0 \) in accordance with the sequence of \( \{v_\alpha(n)\} \) as follows

\[
0, 1, \ldots, l_1 - 1; l_1, l_1 + 1, \ldots, l_1 + l_2 - 1; \ldots; \sum_{i=1}^{T(\alpha)-1} l_i, \ldots, \sum_{i=1}^{T(\alpha)} l_i - 1,
\]

\[
\sum_{i=1}^{T(\alpha)} l_i, \sum_{i=1}^{T(\alpha)} l_i + 1, \ldots, \sum_{i=1}^{T(\alpha)} l_i + \sum_{j=1}^{T(\alpha)} P_{\nu_\alpha}(T(\alpha) + 1, j)l_j - 1, \ldots \tag{3.12}
\]

Continuing in this way, for a \( k \in \mathbb{N}_0 \), we obtain a formal representation with respect to the sequence \( \{v_\alpha(n)\} \) with an aperiodic behavior:

\[
\begin{align*}
k &= k_0 + \sum_{i=1}^{m} l_i + \sum_{i=1}^{nT(\alpha)} P_{\nu_\alpha}(i, j)l_j \\
&+ \sum_{i=1}^{m} \sum_{j=1}^{T(\alpha)} P_{\nu_\alpha}\left(n(k)T(\alpha) + i, j\right)l_j + k_1, \\
m &= 0, 1, \ldots, T(\alpha) - 1; \ k_0 = 0, 1, \ldots, l_{i+1} - 1, \\
n &= 0, 1, \ldots, T(\alpha) - 1; \ k_1 = \sum_{j=1}^{T(\alpha)} P_{\nu_\alpha}\left(n(k)T(\alpha) + n + 1, j\right)l_j - 1, \\
n &= 0, 1, \ldots \tag{3.13}
\end{align*}
\]

In view of the representation (3.13), the statement below is immediate.

Theorem 3.2. Suppose \( \alpha \in \mathbb{Q}^c \), then

\[
\lim_{k \to +\infty} \frac{1}{n(k)T(\alpha)} \sum_{i=1}^{n(k)T(\alpha)} \sum_{j=1}^{T(\alpha)} P_{\nu_\alpha}(i, j)l_j = \frac{1}{\alpha}. \tag{3.14}
\]
We summarize the above discussions in the following statement.

**Theorem 3.3.** Suppose \( f : \mathbb{N}_0 \to \mathbb{N}_0 \) is an unbounded non-monotone and onto function and \( g : \mathbb{N} \to \mathbb{N}_0 \) is a bounded function defined by \( g(n) = |f^{-1}(n - 1)|. \) Then \( \mathbb{N}_0 \) has a formal representation with respect to \( g, \) in the sense below

\[
\begin{align*}
  k &= k_0 + \sum_{i=1}^{m} g(i) + \sum_{j=1}^{n(k)T} P_g(i,j)g(j) \\
  &\quad + \sum_{j=1}^{n(k)} P_g(n(k)T + i,j)g(j) + k_1, \\
  m &= 0, 1, \ldots, T - 1; \quad k_0 = 0, 1, \ldots, g(m + 1) - 1, \\
  n &= 0, 1, \ldots, T - 1; \quad k_1 = 0, 1, \ldots, \sum_{j=1}^{T} P_g(n(k)T + n + 1,j)g(j) - 1, \\
  n(k) &= 0, 1, 2, \ldots,
\end{align*}
\]

where

\[
T = \min \left\{ i \in \mathbb{N}_0 \mid g(j) \in \{g(1), g(2), \ldots, g(i)\}, \quad j \in \mathbb{N}_{i+1} \right\}.
\]

and

\[
P_g(i,j) = \begin{cases} 
1, & \text{ith entry of the sequence } g(\mathbb{N}) \text{ is } g(j) \\
0, & \text{otherwise}
\end{cases}
\]

Moreover, if

\[
\lim_{k \to +\infty} \frac{f(k)}{k} = \lambda_0 \neq 0,
\]

then

\[
\lim_{k \to +\infty} \sum_{i=1}^{n(k)T} \sum_{j=1}^{T} P_g(i,j) \frac{g(j)}{f(k)} = \frac{1}{\lambda_0}.
\]

Suppose \( g(\mathbb{N}) \) is periodic with period \( T. \) Let \( t : \{1, 2, \ldots, T\} \to \{1, 2, \ldots, T\} \) be defined by

\[
t(j) = \min \left\{ i \in \{1, 2, \ldots, T\} \mid g(j) = g(i) \right\}.
\]

Let \( S = \max t(j) \) and \( r_i \) denote \( \sharp \{g(i) \mid g(j) = g(i) \}, \quad i = 1, 2, \ldots, T. \) Then \( \sum_{i=1}^{S} r_i = T \) since \( r_i \) is the number of occurrences for \( g(i) \) in each \( T \)-cycle. Moreover, the periodicity implies that

\[
\sum_{i=1}^{n(k)T} P_g(i,j) = n(k) \sum_{i=1}^{T} P_g(i,j)
\]

\[
= n(k) r_{t(j)}.
\]

**Remark 3.1.** Consider the system (1.5) where \( F \) satisfies the existence and uniqueness conditions and \( f \) is defined as in Theorem 3.3. Then \( n(k) \) in representation (3.15) is the number of cycles including different steps of lengths \( g(j)s \) for which we need to go back and utilize the previous states in order to obtain \( u(k) \).

For a periodic \( g, \) the following asymptotic relation holds for \( n(k) \).

**Theorem 3.4.** Suppose \( g \) is \( T \)-periodic and assume that (3.18) holds. Then for \( n(k) \) which appears in the formal representation (3.15), we have

\[
n(k) \sim \frac{k}{\rho_0} + o(f(k)), \quad \text{as } k \to \infty
\]

where \( \rho_0 = \sum_{j=1}^{T} g(j) r_{t(j)}. \)

**Proof.** From (3.19) and (3.21) we obtain

\[
\lim_{k \to +\infty} \frac{k}{f(k)} = \lim_{k \to +\infty} \sum_{i=1}^{n(k)T} \sum_{j=1}^{T} P_g(i,j) \frac{g(j)}{f(k)}
\]

\[
= \sum_{j=1}^{T} g(j) \left( \lim_{k \to +\infty} \frac{1}{f(k)} \sum_{i=1}^{n(k)T} P_g(i,j) \right)
\]

\[
= \sum_{j=1}^{T} g(j) \frac{1}{\lambda_0}.
\]

\[
\frac{k}{f(k)} \sim \frac{1}{\lambda_0} \quad \text{as } k \to \infty.
\]
\[
\sum_{j=1}^{T} g(j) \left( \lim_{k \to +\infty} \frac{n(k)}{f(k)} \sum_{i=1}^{T} p_i(i, j) \right)
\]
\[
= \rho_0 \lim_{k \to +\infty} \frac{n(k)}{f(k)}.
\]
Therefore
\[
\rho_0 n(k) - k = o(f(k)), \quad \text{as } k \to \infty
\]
and it completes the proof. \(\square\)

4. Conclusion

In this work, a class of retarded equations involving unbounded delay is studied. The equations were inspired by a class of singular integral equations on \(q\)-timescales which possess a kernel with compact support. An exponential global stability result is obtained then dynamics of the delay dependence of the solution is investigated.

Taking advantage of a formal representation for natural numbers, an asymptotic behavior of the backward dependence of a solution to an extended class of retarded equations with unbounded delay is obtained.

References