1 Introduction

Derivative on fractals (as a special case of time scales) has been proposed in Ref. [1] by

\[ D^\alpha f(t) := \lim_{\tau \to \alpha} \frac{f(t) - f(t')}{t^\alpha - (t')^\alpha} \]  

and it is named as Hausdorff derivative. This definition is a reminiscent of the concept of Hölder exponent and self-similarity. The proposed derivative is given in such a way that it is in accord with the Hausdorff dimension of a fractal.

Calculus on time scales has been introduced by Hilger in his Ph.D. thesis and upon which difference and differential calculus is unified [2,3]. Advances in this theory were achieved by the contribution of some mathematicians, such as Hilger, Bohner, Peterson, Agarwal, and others.

Recently in Ref. [4], authors proposed a new derivative on time scales (in the sense of Hilger) which is similar to Ref. [1] and they enriched the paper with some illustrative examples involving the Cantor set. A calculus in any way it is defined as that it must have a rigorous definition of derivative and its inverse phenomenon, namely, the integral. These two main parts of the newly born calculus must support each other in such a way that it will be applicable in dynamic equations. So, one of our goals is to discuss the indefinite integral. These two main parts of the newly born calculus constitute a reasonable interdisciplinary research area [5–9].

Burger’s equation is one of the fundamental partial differential equations, which occurs in various fields, such as fluid mechanics, gas dynamics, and traffic flow. In this paper, a traffic flow problem is described by an initial value problem involving a fractional Burger’s equation on a time scale.

This paper is organized as follows: Section 2 is devoted to some preliminaries on \( \Delta \)-calculus and fractional calculus on time scales. In Sec. 3, by relaxing the sufficient conditions of the fractional chain rule theorem (Theorem 19 [4]), which is a fractional generalization of Theorem 1.87 [10], another chain rule is derived. In Sec. 4, we discuss the definition of the fractional indefinite integral that is derived from Definition 24 [4]. Section 5 deals with a traffic flow problem which is an initial value problem involving Burger’s equation.

2 \( \Delta \)-Calculus and Fractional Calculus on Time Scales

For more detailed discussions on the calculus on time scales, see Refs. [2–4] and [10].

Time scales are closed subsets of the real line. They may appear as the sets of closed intervals or any scattered points. The most important examples are \( \mathbb{Z} \), \( h \mathbb{Z} \) (\( h > 0 \)), \( 2^\mathbb{Z} \).

The shift forward and shift backward functions (denoted by \( \sigma \) and \( \rho \), respectively) are defined by

\[ \sigma(t) := \inf \{ s \mid s > t \} \]  

\[ \rho(t) := \sup \{ s \mid s < t \} \]

The graininess function is defined by

\[ \mu(t) := \sigma(t) - t \]

The point \( t \in T \) is said to be right-scattered, if \( \mu(t) > 0 \), and it is called right-dense, if \( \mu(t) = 0 \).

If \( T \) has a left-scattered maximum \( t_0 \), then \( T^x \) is defined by \( T \setminus \{ t_0 \} \).

The counterpart of continuity in the calculus on time scales is the rd-(right-dense) continuity. A function \( f: \mathbb{T}^x \to \mathbb{R} \) is said to be rd-continuous, if it is continuous at right-dense points, and the left-side limit on left-dense points exists. The set of such functions is denoted by \( C_{rd}(\mathbb{T}) \) or simply \( C_{rd} \).

The shift forward and backward operators \( \sigma_\rho: C_{rd}(\mathbb{T}) \to C_{rd}(\mathbb{T}^x) \) can be defined by \( \sigma_\rho f(x) = f(\sigma(x)) \) and \( \rho_\sigma f(x) = f(\rho(x)) \).
Let $T = h\mathbb{Z}$ with $h \in \mathbb{R}^+$. Then, $\sigma(t) = t + h$ and $\rho(t) = t - h$. For a constant function $f \equiv c$ on $T$, we have $\sigma(f) = f$ and $\rho(f) = f$; therefore

$$\sigma I f = (\rho I f) = f = 0 \quad (5)$$

In other words, both $\sigma - I$ and $\rho - I$ are invertible. In general, for an arbitrary time scale $T$ with or without jump, any constant function $f \equiv c$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$ of both operators $\sigma$ and $\rho$. So both $\sigma - I$ and $\rho - I$ are invertible.

Since the calculus on time scales unifies difference and differential, the symbol $\Delta$ which used to be the symbol of the difference is still being used as the symbol of the derivative.

**Definition 2.1.** Suppose $f : T \to \mathbb{R}$ is a function and $t \in T^k$. If $f$ is continuous at $t$ and $t$ is right-scattered, then $f$ is differentiable at $t$ with $f^\Delta(t) = [f(\sigma(t)) - f(t)]/\mu(t)$. If $t$ is right-dense, then $f$ is differentiable at $t$ if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,$f^\Delta(t) := \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$.

**Theorem 2.2.** ([10]) Assume that $f : T \to \mathbb{R}$ is a function and let $t \in T^k$. Then, we have the following:

1. If $f$ is differentiable at $t$, then $f$ is continuous at $t$.
2. If $f$ is continuous at $t$ and $t$ is right-scattered, then $f$ is differentiable at $t$ with $f^\Delta(t) = [f(\sigma(t)) - f(t)]/\mu(t)$.
3. If $t$ is right-dense, then $f$ is differentiable at $t$ if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,$f^\Delta(t) := \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$.

4. If $f$ is differentiable at $t$, then $f(\sigma(t)) = f(t) + \mu(t)^2 f^\Delta(t)$.

In parallel to the derivative on time scales, we have the next definition.

**Definition 2.3.** ([4]) Let $f : T \to \mathbb{R}$ be a function, $t \in T^k$ and $\alpha \in \{1/q \mid q \text{ is odd}\}$. We define $f^\alpha(t)$ to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood $U$ of $t$ so that

$$|f(\sigma(t)) - f(s) - (\sigma(t) - s)^\alpha f^\alpha(t)| \leq \varepsilon |\sigma(t) - s|^\alpha \quad (7)$$

if $f$ is continuous at $t$ and $t$ is right-scattered, then $f$ is differentiable of order $\alpha$ at $t$ with

$$f^\alpha(t) = \frac{f(\sigma(t)) - f(t)}{\mu^\alpha(t)} \quad (8)$$

If $t$ is left-dense, then $f$ is the fractional differential of order $\alpha$ at $t$ if and only if

$$\lim_{s \to t^-} \frac{f(t) - f(s)}{t - s}$$

exists and then it will be $f^{\alpha^\Delta}(t)$.

It is worth mentioning that, in Definition 2.3, to avoid the complex values, one can assume that $\alpha$ belongs to the set $(0, 1] \cap \{1/2m + 1 \mid n, m \in \mathbb{N}\}$. However, there will be no concern to relax $\alpha$ to the interval $(0, 1]$, since in the remaining parts of this paper, points are right-scattered (see Definition 6 and Theorem 7 [4]).

One can define the fractional differential operator on time scale and denote it by $D^\alpha$.

Almost all expected rules, such as linearity, product rule, and chain rule, are hold, since we are dealing with a local derivative. For example, for two functions $f$ and $g$, (provided their derivatives exist) we have

$$(f(t)g(t))^\Delta = f^\Delta(t) g(t) + f(\sigma(t))g^\Delta(t) \quad (10)$$

By the existence of antiderivative theorem [3], for every rd-continuous function $f : T \to \mathbb{R}$, there is a differentiable function $F$ with

$$F^\Delta(t) = f(t), \quad \forall t \in T^k \quad (11)$$

and we denote it by

$$\int_t^s f(t) \mu(t) \, dt = F(t) - F(s) \quad (12)$$

Also, we have the following theorem (Theorem 1.75 [10]):

**Theorem 2.4.** ([10]) Let $f \in C_{\alpha}(T)$.

(a) For $t \in T^k$, we have $f^{(0)}(t) = f(t)$. (b) If $[a, b] \cap T$ consists of only isolated points, then

$$\int_t^s f(t) \Delta t = \sum_{t_0 \in [a, b] \cap T} f(t_0) \mu(t).$$

**3 Fractional Chain Rule**

Chain rule is given by Theorem 19 in Ref. [4], that is

**Theorem 3.1.** (chain rule) Let $z \in (0, 1]$. Assume that $g : T \to \mathbb{R}$ is continuous, $\alpha : T \to \mathbb{R}$ is fractional differentiable of order $\alpha$ at $t \in T^k$, and $f : T \to \mathbb{R}$ is continuously differentiable. Then, there exists $c$ in the real interval $[t, \sigma(t)]$ with

$$f^{(0)}(g(t)) = f'(c) g'(t) \quad (13)$$

It is worth mentioning that under the assumption of continuous differentiability of $f$, there is another chain rule due to Pötzschke [2].

Regarding the importance of the chain rule in the dynamic equations, by relaxing the conditions on $f$ and $g$, some modification to Theorem 1 will be done, so that it will be more applicable.

For the next purposes, we need the following definitions:

$$T_1 := \{t \in T \mid \mu(t) \neq 0\}, \quad T_2 := \{t \in T \mid \mu(t) = 0\}$$

$$J := \{f \in C_{\alpha}(T) \mid \forall t \in T_1, f(\sigma(t)) \neq f(t)\}$$

**Theorem 3.2.** ([10]). Assume that $\nu : T \to \mathbb{R}$ is strictly increasing and $\Delta = \Delta(T)$ is a time scale. Let $\omega : T \to \mathbb{R}$. If $\nu^\Delta(t)$ and $\omega^\Delta(t)$ exist for $t \in T^k$, then

$$(\alpha \circ \nu)^\Delta(t) = (\omega^\Delta(t) \circ \nu)^\Delta(t) \quad (16)$$

The same result is obtained if $\nu$ satisfies the sufficient condition to be strictly increasing.

**Lemma 3.3.** For a given function $g : T \to \mathbb{R}$,

(a) if $g$ is increasing, then $g(\sigma(t)) = g(\sigma(t))$. (b) If $g(\sigma(t)) = g(\sigma(t))$ and $g \in J$, then $g(t) < g(\sigma(t))$.

**Proof.** (a) In contrast, we assume that $g(t)$ is not the predecessor of $g(\sigma(t))$. So there must be some $u \in \mathbb{I}(g)$ (some $u \in T$) with $g(u) < g(\sigma(t))$. If $s < t$ then $u < g(t)$ since $g$ is increasing. Similarly, $s > \sigma(t)$ results in $u > g(\sigma(t))$. Therefore, $s \in (t, \sigma(t)) = \emptyset$ and consequently $g(t)$ is the predecessor of $g(\sigma(t))$. (b) If $g(t) > g(\sigma(t))$, then $g(t) > g(\sigma(t))$ and this contradicts the definition of $\sigma(t)$.

**Corollary 3.4.**

(a) If $g \in J$, then $g(t) < g(\sigma(t))$ iff $g(\sigma(t)) = g(\sigma(t))$. 031022-2 / Vol. 12, MAY 2017 Transactions of the ASME
(b) If \([s_0, t_0] \cap \mathbb{T} \subset \mathbb{T}_1\), then for every \(t \in [s_0, t_0]\), \(g(t) < g(\sigma(t)) \iff g\) is increasing over \([s_0, t_0] \cap \mathbb{T}\).

**Theorem 3.5. (Fractional chain rule)** Let \(g: \mathbb{T} \to \mathbb{R}\) and \(f; g(\mathbb{T}) \to \mathbb{R}\) be such that \(g(\mathbb{T})\) is a time scale and \(g(\sigma(t)) = \sigma(g(t))\). Assume that \(g_s(t), f_s'(g(t)), \) and \(f^s_s(g(t))\) exist for \(t \in \mathbb{T}^\circ\). Then
\[
(f \circ g)^{(s)}(t) = (f' \circ g)(t)g^{s}(t)\tag{17}
\]

Note that, Eq. (17) in case \(\mu(t) = 0\) and \(x \in \{1/q|q\text{ is odd}\}\) is nothing but the following equation:
\[
(f \circ g)^{(s)}(t) = (f' \circ g)(t)g^{s}(t)\tag{18}
\]

**Proof.** For a given \(0 < \varepsilon < 1\), let \(\varepsilon := \varepsilon \left[1 + |g^{(s)}(t)| + |f^{(s)}(g(t))|\right]^{-1}\). The assumptions on \(f\) and \(g\) guarantee the existence of two neighborhoods \(N_1\) and \(N_2\) of the points \(t\) and \(g(t)\), respectively, together with
\[
|g(\sigma(t)) - g(s) - (\sigma(t) - s)^{g}(t)| \leq \varepsilon'|\sigma(t) - s|^s, \quad s \in N_1
\]
and
\[
|f(g(\sigma(t))) - f(g(s)) - [g(\sigma(t)) - g(s)]f^{A}(g(t))| \leq \varepsilon'|g(\sigma(t)) - g(s)|, \quad s \in N_2
\]
For \(s \in N_1 \cap g^{-1}(N_2),\) we have
\[
|f(g(\sigma(t))) - f(g(t)) - [g(\sigma(t)) - g(s)]f^{A}(g(t))| \leq |f'(\sigma(g(t)))| \cdot |\sigma(t) - s|^s|g^{(s)}(t)| + \varepsilon'|\sigma(t) - s|^s \
\leq \varepsilon'|g(\sigma(t)) - g(s)| \cdot |f^{A}(g(t))| + \varepsilon'|\sigma(t) - s|^s|f^{A}(g(t))| \
\leq \varepsilon'|\sigma(t) - s|^s|\sigma(t) - s|^s\leq \varepsilon'|\sigma(t) - s|^s\tag{19}
\]
This proves the relation (17).

4 Fractional Integral

The fractional indefinite integral is given by Definition 24 at Ref. [4], that is
\[
\int f^{A}s(t) := D^{1-s}(\int f f(t))\tag{20}
\]

The above definition has the implication that
\[
D^{s}\left(\int f^{A}s(t)\right) = D^{s}D^{1-s}\left(\int f f(t)\right) = D^{s}\left(\int f f(t)\right) = f\tag{21}
\]
and the last equality is in the sense of the existence of antiderivative [3].

The aim of this section is to show that the above-mentioned definition is too narrow to include any possible time scale with arbitrary graininess \(\mu\). To this end, we need the following statements:

**Theorem 4.1.** Suppose \(\mathbb{T}\) is a time scale with nonconstant and positive graininess \(\mu\). Then

a) \(D^{s}D^{b} = D^{s+b}\), \(\alpha, b \in (0, 1), \iff\)
\[
\sigma = 2\left(\frac{\mu^{\beta}}{\mu}\right)^{1-s} I
\]
(22)

b) If \(\alpha, b \in (0, 1)\) with \(\alpha \neq b\), then \(D^{s}D^{b} = D^{b}D^{s}\), \iff:
\[
\sigma = \frac{(\mu^{\beta} - \mu^{s})^{\beta}}{\mu^{\beta} - \mu^{s}} I
\]
(23)

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The relation (30) yields that \( \mu(\alpha^{n+1}(0)) = \mu(\alpha^n(0)) \)
\((\alpha^{n+1}(0)/2)^{1/\alpha} \). Let \( a = \sigma(0) \). For the \((n+1)\)th point, an algebraic calculation gives
\[
\mu(x_{n+1}) = a \left( \frac{a}{2} \right)^{1-\alpha} \left[ a + a \left( \frac{a}{2} \right)^{1-\alpha} \right]^{-1/\alpha} \times \cdots \times f(n)
\]
where \( f(n) \) is the \( n \)th factor related to the presence of \( \mu(x(n)) \). For example,
\[
x_0 = 0, \quad x_1 = \sigma(0) = a, \quad x_2 = \sigma^2(0) = a + a \left( \frac{a}{2} \right)^{1-\alpha}
\]
\[
x_3 = \sigma^3(0) = a + a \left( \frac{a}{2} \right)^{1-\alpha} + a \left( \frac{a}{2} \right)^{1-\alpha} \left[ a + a \left( \frac{a}{2} \right)^{1-\alpha} \right]^{-1/\alpha}
\]
According to Hilger [3], for every \( f \in C_{rd}(T) \), there exists some \( g: T \to R \), which satisfies \( g^\Delta = f \) and we denote it by \( g = \int f \Delta t \).
If we assume that \( f/\mu^{1-\alpha} \) be rd-continuous, then there exists an antiderivative, which is
\[
\int f(t) / \mu(t)^{1-\alpha} \Delta t = g(t)
\]
In other words: \( f(t)/\mu(t)^{1-\alpha} = Dg(t) = g(\sigma(t)) - g(t)/\mu(t) \) which yields: \( D^\alpha g(t) = f(t) \) and it can be represented in an integral form by
\[
\int f \Delta^\alpha t = g(t)
\]

5 Fractional Burger’s Equation

One of the classic models in traffic flow theory which is simply derived under idealistic assumptions is the initial value problem involving the Burger’s equation of the form
\[
u(x, t) + (u(x, t)\nu(x, t))_x = 0 \quad (35)
\]
Here, \( u(x, t) \) is the density of cars in the street, and \( \nu(x, t) \) is a velocity function.
In this section, first we suppose a fractional Burger’s equation. Then, without using the indefinite fractional integral which is defined by Eq. (22), a recurrence equation that is equivalent to the Volterra integral equation of the given problem will be derived.
Assume that there are many junctions along a street and each is equipped with a traffic light that forces the traffic flow to stop periodically. The positions of junctions are assumed to be the points of a time scale \( \mathbb{X} \), which starts from 0. Also assume that the time \( t \) ranging over a time scale \( \mathbb{T} \) has to be known. The fractional rate of change to the density of cars behind the red lights of two successive junctions \( x \) and \( \sigma(x) \) is
\[
D^\alpha_t \left( \int_{t_0}^{t_1} u(s, t) \Delta s \right) = f(u(x, t)) - f(u(\sigma(x), t))
\]
Dividing both sides of Eq. (36) by \( \mu(x) \), where \( \mu \) is the graininess function of \( \mathbb{X} \), we get
\[
\frac{1}{\mu(x)} \int_{t_0}^{t_1} u(s, t) \Delta s = u(x, t) = \frac{f(u(x, t)) - f(u(\sigma(x), t))}{\mu(x)}
\]
and we arrive at the following fractional initial value problem:
\[
\begin{cases}
u_t + f(u)_x = 0, \\
u(0, t) = \phi(x)
\end{cases}
\]
This problem is a typical example and one may regard the other variations of Eq. (37) in the presence of the second-order derivative \( u_{xx} \) related to the presence of some parameter \( \nu \), which is called viscosity coefficient.
In the remaining of this work, we suppose the following second-order fractional Burger’s equation:
\[
\begin{cases}
u_t - 2\mu u_{xx} - \mu u^2_{xx} = 0 \\
u(0, t) = \phi(x), \quad u(\alpha, t) = \psi(t)
\end{cases}
\]
where \( u: \mathbb{X} \times \mathbb{T} \to R \). Suppose that \( \mathbb{X} \) has the property for which Eq. (30) holds.
To convert Eq. (38) into an ordinary differential equation in the sense of \( \Delta \)-derivative, by making use of Eq. (30) and the product rule for the fractional derivative on time scales [4], we have
\[
\begin{align}
\nu^{(2\alpha)}(x)_x &= \nu^{(\alpha)}(x)_x = \mu^{1-\alpha}(x) \left( \mu(x)^{1-\alpha} u^\Delta_x (x, t) \right)_x = \mu^{1-\alpha}(x) \left( \mu(x)^{1-\alpha} u^\Delta_x (x, t) \right) \\
&= \mu^{1-\alpha}(x) \left( \mu(x)^{1-\alpha} - \mu^{1-\alpha}(x)^{1-\alpha} u^\Delta_x (x, t) + \mu^{1-\alpha}(x)^{1-\alpha} \sigma(x)_x^{1-\alpha} u^\Delta_x (x, t) \right) \\
&= \mu^{1-\alpha}(x) \left( \mu^{1-\alpha}(x)^{1-\alpha} - \mu^{1-\alpha}(x)^{1-\alpha} u^\Delta_x (x, t) + \mu^{1-\alpha}(x)^{1-\alpha} \sigma(x)_x^{1-\alpha} u^\Delta_x (x, t) \right)
\end{align}
\]
Equation (38) becomes
\[
\nu^{(\alpha)}(x)_x = \left( \frac{2\mu^{1-\alpha} u + \mu^{1-\alpha} \sigma}{2} \right) u_\Delta - \mu^{2(1-\alpha)} \frac{\sigma}{2} u_{xx} = 0
\]

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and after rearranging, the initial value problem \((38)\) will be

\[
\begin{align*}
\mu^{\Delta \Delta} &= \frac{2}{\mu^{2(1-x)\sigma} u_x^{(s)}} - \frac{4 \mu^2 u + \sigma}{\mu^2} u_x^{(s)} \\
\mu(0, t) &= \phi(t), \quad u(a, t) = \psi(t)
\end{align*}
\]

\[(41)\]

**Proposition 5.1.** Initial value problem \((41)\) is equivalent to the following recurrent equation:

\[
u(x_{n-1}, t) \left(1 - \frac{4 \mu^2(x_n) u(x_n, t) + \sigma(x_n)}{\mu(x_n) \sigma(x_n)}\right) = \phi(t) + x_{n+1} \psi(t)
\]

\[
+ \sum_{i=0}^{n-1} \left(x_{i+1} - x_i\right) \left( \frac{2u_x^{(s)}(x_i, t)}{\mu^{1-2x_i}(x_i) \sigma(x_i)} - \frac{4 \mu^2(x_i) u(x_i, t) + \sigma(x_i)}{\mu(x_i) \sigma(x_i)} \left(u(x_{i+1}, t) - u(x_i, t)\right) \right)
\]

\[
+ \left(x_{n+1} - x_n\right) \left( \frac{2u_x^{(s)}(x_n, t)}{\mu^{1-2x_n}(x_n) \sigma(x_n)} - \frac{4 \mu^2(x_n) u(x_n, t) + \sigma(x_n)}{\mu(x_n) \sigma(x_n)} u(x_n, t) \right)
\]

\[(42)\]

\[
\text{together with the initial conditions } u(0, t) = \phi(t) \text{ and } u(a, t) = \psi(t).
\]

**Proof.** By double integration of both sides of Eq. \((41)\) and making use of the following equality:

\[
\int_0^t \int_0^s f(\tau) d\tau = \int_0^t (x - s)f(s) \Delta s
\]

\[(43)\]

one obtains

\[
u(x, t) = \phi(t) + \psi(t) + \int_0^t \frac{2(x - s)u_x^{(s)}(s, t) \Delta s}{\mu^{2(1-x)}(s) \sigma(s)} - \int_0^t \frac{(x - s) [4 \mu^2(s) u(s, t) + \sigma(s)] u_x^{(s)}(s, t) \Delta s}{\mu(s) \sigma(s)}
\]

\[(44)\]

Utilizing Theorem 2.4, that is

\[
\int_0^{x_{n+1}} f(\tau) \Delta \tau = \sum_{i=0}^{n} \mu(x_i) f(x_i) = \sum_{i=0}^{n} \mu(x_i) f(x_i)
\]

\[(45)\]

\[
\text{together with Eq. \((30)\) and } \mu(0) = \sigma(0) = a, \text{ we obtain}
\]

\[
u(x_{n+1}, t) = \phi(t) + \psi(t) + \sum_{i=0}^{n-1} \left(x_{i+1} - x_i\right) \left( \frac{2u_x^{(s)}(x_i, t)}{\mu^{1-2x_i}(x_i) \sigma(x_i)} - \frac{4 \mu^2(x_i) u(x_i, t) + \sigma(x_i)}{\mu(x_i) \sigma(x_i)} \left(u(x_{i+1}, t) - u(x_i, t)\right) \right)
\]

\[
+ \left(x_{n+1} - x_n\right) \left( \frac{2u_x^{(s)}(x_n, t)}{\mu^{1-2x_n}(x_n) \sigma(x_n)} - \frac{4 \mu^2(x_n) u(x_n, t) + \sigma(x_n)}{\mu(x_n) \sigma(x_n)} u(x_n, t) \right)
\]

\[(46)\]

**Table 1** Different densities at the fourth traffic light for types of vehicles: \(u(x, 0)\) with \(a = 1.001\) and \(x = 0.75\)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Business day track</td>
<td>144.32</td>
<td>302.761</td>
<td>427.498</td>
<td>366.379</td>
<td>240.659</td>
<td>202.418</td>
</tr>
<tr>
<td>Urban cars</td>
<td>90.1298</td>
<td>267.131</td>
<td>438.963</td>
<td>394.113</td>
<td>274.63</td>
<td>101.162</td>
</tr>
<tr>
<td>Through truck</td>
<td>333.635</td>
<td>332.379</td>
<td>325.449</td>
<td>318.936</td>
<td>339.242</td>
<td>335.017</td>
</tr>
</tbody>
</table>

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Example 5.2. Let \( T = N_0 \). The initial function \( \phi(t) = u(0, t) \) is derived by interpolating a real sample data pertaining to the density of four types vehicles during 24 hrs in Vancouver, Canada. Also, let \( \psi(t) = u(a, t) = a \phi(t) \) with \( a = 1.001 \). For \( a = 0.75 \), four densities at \( x_4 \) (fourth traffic light) pertaining to four types of vehicles are given (Table 1 and Fig. 1).

For rural cars, we have the following \( \phi(t) \):

\[
\phi(t) = 0.000324568r^5 + 0.0264435r^4 - 2.13015r^3 + 38.5188r^2 - 173.996r + 275.524
\]

there is \( \phi(t) \) for business day track

\[
\phi(t) = -0.00310692t^2 + 0.245294t^3 - 6.85614t^4 + 77.2953t^5 - 274.368t + 413.269
\]  

The following \( \phi(t) \) is for urban cars:

\[
\phi(t) = -0.00108005t^5 + 0.102533t^4 - 3.36463t^3 + 42.0473t^2 - 133.147t + 200.264
\]

For through truck, we let the following \( \phi(t) \):

\[
\phi(t) = -0.000596146t^5 + 0.0365573t^4 - 0.783261t^3 + 7.13108t^2 - 28.2998t + 451.306
\]  

References