On a Discrete Chaos Induction Via an Aperiodic Kicks Pattern

In this work, a class of kicked systems perturbed with an irregular kicks pattern is studied and formation of a chaos in the senses of Devaney and Li–Yorke in the corresponding discretized system is investigated. Beside a discussion on chaotic stability, an example is presented. Then, the existence of a period three orbit of a 2D map which governs a class of dynamic problems on time scales is studied. As an application, a chaotic encryption scheme for a time-dependent plain text with the help of chaos induction in the sense of Li–Yorke is presented. [DOI: 10.1115/1.4035078]

Keywords: kicked Hamiltonian systems, Devaney chaos, Li–Yorke chaos, chaotic stability, dynamics on time scales, chaos anticontrol, communication systems, chaotic encryption

1 Introduction

Hamiltonian mechanics, which was a continuation of an effort started by Lagrangian mechanics, was trying to formulate a broad class of observed phenomena. The word formulation sometimes has an implication to the prediction of the future of a system and it is postulated that, in an ideal or maybe an illusional case, this prediction is possible provided the system is integrable. As it was observed in abundant natural experiences (if not for almost all cases), the existence of irregularity caused by sensitivity to the initial conditions [1].

Despite the lack of predictability in chaotic systems, one of the interesting applications of chaotic nonlinear system is in communication systems as well as electromagnetic interference reduction in some areas as the Dirac generalized function and $\delta(t)$ as the Dirac generalized function and $T = 2\pi/\nu$ the amplitude of the pulses. Utilizing

$$H = H_0(p) + Kf(x) \sum_{k=-\infty}^{\infty} \delta\left(\frac{t}{T} - k\right)$$ (1.1)

with $\delta(t)$ as the Dirac generalized function and $T = 2\pi/\nu$ the amplitude of the pulses. Utilizing

$$\frac{\partial H}{\partial p} = Dq(t), \quad \frac{\partial H}{\partial q} = -Dp(t)$$ (1.2)

with identifying the generalized coordinate $q$ and momentum $p$, respectively, by $x$ and $p$ (in which, the mass $m$ is normalized to write $Dx(t) = p(t)$), one obtain the following system:

$$\begin{cases}
Dx(t) = H_0'(p) = o(x(p) \\
Dp(t) = -Kf'(x) \sum_{k=-\infty}^{\infty} \delta\left(\frac{t}{T} - k\right)
\end{cases}$$ (1.3)

Periodically perturbed systems such as kicked rotator are extensively studied both numerically and topologically [6–11]. Also, some of the recent works were dealt with study of the chaotic behavior of the noninteger order kicked systems [12,13]. It must be noted that, in both of the above mentioned classes of studies, the kicks patterns are presumed to be of constant period.

In this work, letting kicks to perturb the system arbitrarily, i.e., relaxing the kick’s time by introducing $\{t_n\}_n$ with $t_0 > 0$ and $\sigma(t_k) = t_{k+1}$, we study the system proposed by

$$\begin{cases}
Dx(t) = p(t) \\
Dp(t) = g(x(t)) + f(x(t)) \sum_{k=0}^{\infty} \delta(t - \sigma(t_k)) \\
x(0) = x_0, \quad p(0) = p_0
\end{cases}$$ (1.4)

where $f, g : \mathbb{R} \to \mathbb{R}$ are continuously differentiable functions with their further properties to be known later. Note that the function $\sigma$ is supposed to be the usual shift forward operator on the presumed time scale $T = \text{closure}\{t_n|n \in \mathbb{N}_0\}$. [JCDN] 041008-1
Closed subsets of the real line are called time scales. In theory or application, we may come across to a class of mathematical objects including a union of closed or compact intervals, lattices, some complicate cluster of points such as Cantor set, or some fractals. Calculus on time scales covers the $q$-calculus, difference calculus, and classical calculus as well (see Ref. [14] for a short review and Ref. [15] for perfect discussion).

The functions $\sigma$, $\rho$, and $\mu$ are defined by

$$\sigma(t) := \inf \{s \in T | s > t \}, \quad \rho(t) := \sup \{s \in T | s < t \}, \quad \mu(t) := \sigma(t) - t$$

(1.5)

A right-scattered (right dense) point $t \in T$ is defined by $\mu(t) > 0$ ($\mu(t) = 0$). If $\max T \neq \emptyset$, then we denote $T^s = T \setminus \max T$, otherwise $T^s = T$.

**Definition 1.1** [15]. For a function $f : T \rightarrow \mathbb{R}$ and $t \in T^s$, we define $\Delta f(t)$ to be a number (provided it exists) such that, for an $\epsilon > 0$, a neighborhood of $t$ (i.e., $U = (t - \delta, t + \delta)$ for some $\delta > 0$) exists so that

$$|f(\sigma(t)) - f(s) - (\sigma(t) - s)\Delta f(t)| \leq \epsilon |\sigma(t) - s|$$

(1.6)

A useful formula holds when $f$ is differentiable

$$f(\sigma(t)) = f(t) + \mu(t)\Delta f(t)$$

(1.7)

As it will be observed, deriving a discrete map, which is called the universal map from the system (1.4), facilitates the study of its qualitative behavior. To obtain the universal map, regarding the early dynamic of $\sigma$, the system (1.4) with the notation of $\Lambda$-derivative on time scale $\mathbb{T}$ can be converted to a first-order singular system ($t \in \mathbb{T}$)

$$\begin{cases}
    x(t) = p(t) + f(x(t)) + 2\mu(t)g(x(t)) \\
    p(t) = \Delta g(x(t)) / \mu(t)
\end{cases}$$

(1.10)

while we utilize the estimation

$$\int_{t_n}^{t_{n+1}} (t_{n+1} - s) g(x(s)) \, ds \simeq \int_{t_n}^{t_{n+1}} (t_{n+1} - s) g(x(t_n)) \Delta$$

$$= \mu^2(t_n) g(x(t_n))$$

(1.11)

If we further assume that $H(\mu_n) = \mu_{n+1}$, for some $H$ (it will be introduced later), then Eq. (1.10) turns out to be a map of the form

$$F[x_n, p_n, \mu_n] = \left( x_n + \mu_n p_n + \mu_n^2 g(x_n) + 2\mu_n^2 g(x_n) \right)$$

(1.11)

Remark 1.1. The system (1.11) that is derived by exploiting error estimation will be studied as an approximation to the discretized system (2.12) coupled with Eq. (1.4). In Sec. 2.4, it will be shown that under quite smoothness of $g$ with some given conditions, chaos detection in the approximated system (1.11) ensures the same chaotic dynamic for Eq. (2.12), which is derived from an exact (nonapproximative) discretization of the original system (1.4).

**Example 1.1.** Letting $g(x) = 0, f(x) = -K \sin(x)$, and $T = N_0$ or $Z$, we obtain the standard map (or Chirikov-Taylor map)

$$(x_{n+1}, p_{n+1}) = (x_n + p_{n+1}, p_n - K \sin(x_n))$$

(1.12)

**2 Chaos Detection**

In this section, our aim is to show that, under quite general conditions on the functions $f$ and $g$, the system (1.11) represents a chaotic behavior inherited by an irregular pattern of kicks even if the original system (1.4) is not chaotic with or without period kicks. First, we need to brief on chaos preliminaries.

**2.1 Preliminaries.** One of the simplest dynamical system which involves chaos is the quadratic map $h : [0, 1] \rightarrow \mathbb{R}$ given by the function $h(t) = a(1 - t)$ where $a > 2 + \sqrt{5}$. Although, it has been proved that, for $a = 4$, $h$ involves a Devaney chaos on the interval $[0, 1]$ [17]. For the specific range $a > 2 + \sqrt{5}$, an observation reveals more interesting dynamics enumerated below [17]:

(a) There is a Cantor set $\Lambda \subset [0, 1]$ which is invariant under $h$ if $X$ is a topological space with a topology $\tau$, a subset $\Lambda \subset X$ is Cantor set, if it is closed, perfect, and totally disconnected with respect to $\tau$.

(b) $h$ involves Devaney chaos on $\Lambda$, i.e., it is transitive (topologically) and $\cup_{h} \text{Per}_k(h) = \Lambda$, ($\text{Per}_k(h)$ is the set of points $p$ with $h^k(p) = p$ and $h(p) \neq p$ for $0 \leq k < n$)

(c) There exists some $x \in \Lambda$ for which $\cup_{h} h^k(x) = \Lambda$.

(d) The quadratic map $h : \Lambda \rightarrow \Lambda$ and the shift map $\sigma : \Sigma_2 \rightarrow \Sigma_2$ are topologically conjugate, i.e., there is a homeomorphism $\phi : \Lambda \rightarrow \Sigma_2$ for which $\phi \circ h = \sigma \circ \phi$.

(e) The cardinality of $\text{Per}_n(h)$ (periodic points of order $n$) is $2^n$.

According to the properties (a) – (d), the Devaney chaos is defined below [17]:

**Definition 2.1.** For a set $U$ and a given map $h : U \rightarrow U$ with the set of periodic points denoted by $P \subset U$, we say $h$ is chaotic in the sense of Devaney if it possesses the following properties

(1) $h$ is sensitive to initial conditions.

(2) $h$ has a topological transitivity.

(3) $P$ is dense in $U$.

It is worth to mention that, according to the original definition of chaos in Devaney’s popular book, beside the above conditions in (b), $h$ should be sensitive to initial condition (see Ref. [17], Sec. 1.8); however, Banks et al. in Ref. [18] few years later proved that if a map is transitive and the set of periodic points is dense in an invariant subset of the compact domain, then it is sensitive to initial conditions.

The correspondence between $\Sigma_2$ and $\Lambda$ will be understood by regarding the early dynamic of $h$ upon which there will be two closed intervals $I_0$ and $I_1$ with the property that $\Lambda \subset I_0 \cup I_1$, $1 \leq h^r(t) (t \in I_0)$ and $h^r(t) \leq 1$ ($t \in I_1$). Obviously, $I_0 = [0, a - 1/2a]$ and $I_1 = [a + 1/2a, 1]$. Note that two hyperbolic fixed points $0$ and $a = 1/4$ belong to $I_0$ and $I_1$, respectively.

The conjugacy map $\phi : \Lambda \rightarrow \Sigma_2$ noted in (d) is defined by $\phi(x) = (x_0, x_1, x_2, \ldots) \in \Lambda$ where $x_0 = 0$ if $h^r(x) \in I_0$ and $x_0 = 1$ if $h^r(x) \in I_1$. For example, $(0, 0, \ldots)$ and $(1, 1, \ldots)$ corresponded to $0$ and $a$, and $(0, 1, 1, \ldots)$ corresponded to $h^{-1}(\mu)$.

**Lemma 2.1.** Suppose $h$ is the quadratic map with $a > 2 + \sqrt{5}$. Then, for $m \geq 2$, there exists $x \in I_1$ with $x \neq \mu$ and $h^m(x) = \mu$.

Proof. For $m \geq 2$, the existence of $x$ with the given properties will be clear by observing its counterpart in $\Lambda$ introduced by $s = \frac{1}{m-1}$... and the conjugacy.
Remark 2.1. According to the Li–Yorke’s formulation of chaos in Ref. [19] for a given function \( f \in C^1(J,J) \) (\( J \) is compact), the sufficient condition is to possesses a period three point, i.e., a point \( p \in J \) for which \( f^3(p) = p \) and \( f^j(p) \neq p \) for \( j = 1,2 \). This criterion leads to possessing periodic points of arbitrary orders and also it excludes the possibility for asymptotic periodicity. Apparently, for the classic example \( h \), in view of the conjugacy \( \phi \), having periodic points of arbitrary order is satisfied, that is to say, the quadratic map \( h \) is chaotic in the sense of Li–Yorke. For details on relationships between different types of chaos, see Refs. [20,21].

Definition 2.2 [22]. Let \( X \) be a complete metric space with metric \( d \) and \( f : X \to X \) be a map.

(a) We say that \( z \in X \) is a repeller fixed point of \( f \) in \( \overline{B}_r(z) \) for some \( r > 0 \) or, \( f \) is expanding in \( \overline{B}_r(z) \) and \( f(z) = z \) if the following inequality holds:

\[
\lambda d(x,y) \leq d(f(x),f(y)) \quad \forall x,y \in \overline{B}_r(z)
\]

for some \( \lambda > 1 \).

(b) Let \( z \) be a repeller fixed point in \( \overline{B}_r(z) \) for some \( r > 0 \). Then \( z \) is called a snap-back repeller of \( f \), if there exists a point \( x \in \overline{B}_r(z) \) with \( x \neq z \) and \( f^m(x) = z \) for some \( m \in \mathbb{N} \).

2.2 Main Result. We begin with the following lemma, upon which, a chaotic kicks pattern takes the responsibility for a chaotic behavior of the system (1.11).

Lemma 2.2 [23]. Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a map with a fixed point \( z \in \mathbb{R}^n \). Assume that

(a) \( f \in C^1(B_r(z)) \) for some \( r_0 > 0 \), and for all eigenvalues of \( Df(z) \) namely \( \lambda \), we have \( |\lambda| > 1 \), i.e., \( f \) is expanding in \( \overline{B}_r(z) \) for some \( r > 0 \) with respect to a norm \( || \cdot || \) in \( \mathbb{R}^n \).

(b) \( z \) is a snap-back repeller of \( f \) with \( f^m(x_0) = z \), \( x_0 \neq z \), for some \( x_0 \in B_r(z) \) and some \( m \in \mathbb{N} \). Moreover, the map \( f \) is class \( C^1 \) in some neighborhoods of \( x_0, x_1, \ldots, x_{m-1} \), respectively, and that \( Df(x_j) \neq 0 \) for \( 0 \leq j \leq m-1 \), where \( x_j = f(x_{j-1}) \) for \( 1 \leq j \leq m-1 \).

Then for each neighborhood \( U \) of \( z \), there exists \( k \in \mathbb{N} \) with \( k > m \) and a Cantor set \( \Lambda \subset U \) such that \( f^k : \Lambda \to \Lambda \) is conjugate (topologically) to the shift map \( \sigma : \Sigma_2 \to \Sigma_2 \). Consequently, \( f \) involves Devaney’s chaos on \( \Lambda \) and \( f \) is chaotic in the sense of Li–Yorke. Further, there exists an uncountable, perfect, and compact set \( V \subset \mathbb{R}^n \) containing the Cantor set \( \Lambda \) such that \( f(V) = V \) and \( f \) is chaotic on \( V \) in the sense of Devaney.

The main result of this section which is a discrete chaos induction to a Hamiltonian system is given below provided the following hypothesis is satisfied.

Hypothesis 2.1. Let \( x = \phi^{+}(1,0,1,1, \ldots) \) and suppose \( f,g : \mathbb{R} \to \mathbb{R} \) are continuously differentiable and satisfy \( f(0) = g(0) = 0, f'(0) < -(a + 1/a)g'(0) \) and \( g'(0) \not\in \{-1/\sqrt{2} \mid j = 0 \} \) where \( x_j = h'(x_j) \) for \( j = 0,1,2 \).

Theorem 2.1. Let \( H(t) = at(1 - t) \) with \( a > 2 + \sqrt{5} \), then for \( \mu_0 \in \Lambda \) where \( \Lambda \) is the Cantor set coupled with \( H \), the system (1.11) is chaotic in the sense of Devaney and also in the sense of Li–Yorke.

Proof. Clearly, \( z = (0,0,\mu_0) \) is an expansive fixed point of the system. To show this, note that \( \mathcal{F}(z) = z \) and for each of the eigenvalues namely \( \lambda_j \), we have \( |\lambda_j| = |H'(\mu_j)| = |2 - a| > 1 \).

\[
D\mathcal{F}(z) = \begin{pmatrix} 1 + \mu f'(0) + 2\mu g'(0) & \mu & 0 \\ f'(0) + \mu g'(0) & 1 & 0 \\ 0 & 0 & 2 - a \end{pmatrix}
\]

For two remaining eigenvalues, utilizing \( f'(0) < -(a + 1/a)g'(0) \) in the hypothesis together with \( \mu_0 = a - 1/a \), we get
\[
\frac{1}{2} \left( (2 + \mu f'(0) + 2\mu g'(0)) \right) 
\]

and it guarantees the expansiveness of \( z \) for the map \( \mathcal{F} \). For the second part of this proof, observe that \( \det(D\mathcal{F}(x_j)) = h'(x_j)(1 + x_j^2g'(0)) \neq 0 \) since \( x_j \neq 1/2 \) and \( g'(0) \neq -1/\sqrt{2} \) for \( j = 0,1 \).

Remark 2.2. As a consequence of perturbing a continuous Hamiltonian system by kicks in the manner described in Theorem 2.1, the disrupted paths in \( X-P \) phase space involves the complexity too and the impulsive continuous system reflects the same sensitivity to initial values as the discretized system.

2.3 An Example. Consider the kicked system

\[
\begin{align*}
D^2x(t) &= -x(t) + x(t) \sum_{k=1}^{n} \delta(t_0 - t) \\
x(0) &= 0, \quad Dx(0) = 1
\end{align*}
\]

where \( t_0 \in \mathbb{R}^+ \) and \( \delta(t_0) = \delta(t_0) = a(t_0 - 1 + a(t_0 - 1)) \). Obviously, it satisfies the sufficient condition of Theorem 2.1 and therefore it is chaotic. Consequently, it is sensitive to initial conditions with respect to \( \mu_0 = t_0 - t_0 \). For \( \mu_0 = 0 \in \Lambda \), the corresponding solution \( x_1 \) is

\[
x_1(t) = \left( 1 + t_0 \right) x_1^0 - \int_0^t (t-s)x_1(s)ds \quad t \in (t_0, +\infty)
\]

For \( \epsilon \ll 1 \), there exists \( \mu_0 \in (0, \epsilon) \cap \Lambda \) with dense orbit in \( \Lambda \). For a fixed \( \epsilon \) and such \( \mu_0 \), the solution \( x_2 \) to the corresponding system (2.4) is

\[
x_2(t) = \left( 1 + t_0 \right) x_1^0 - \int_0^t (t-s)x_2(s)ds + \sum_{k=1}^{n} x_{2,k} \quad t \in (t_0, t_{n+1})
\]

and for \( n \geq 2 \), letting \( x_{2,n} = x_{2,n}(t) \) \( (j = 1,2) \), we obtain

\[
x_{2,a} - x_{1,a} = \left( \sum_{k=1}^{n} x_{2,k} \right)
\]

so they become far apart from each other. The sensitivity to initial conditions for the discretized system, that is

\[
\begin{align*}
x_{n+1} &= x_0 (1 + \mu_0 - 2\mu_0 a) + \mu_0 p_0 \\
p_{n+1} &= p_0 + x_0 (1 - \mu_0) \\
\mu_{n+1} &= 4.3\mu_0 (1 - \mu_0)
\end{align*}
\]

is shown in Fig. 1.

2.4 Chaotic Stability Issue. What we have done in Sec. 2.2 were to analyze the estimated system (1.11) rather than the original system (1.8) (which is a discretization of the main problem (1.4)). It should be noticed that by chaotic stability, we mean that the original system (1.8) under the estimation (1.10) conditionally represents the same chaotic behavior. To obtain a sufficient condition, we need the following inequality.

Theorem 2.2 [24]. Let \( g \in C[a,b], \) differentiable on \((a,b)\), and \( g'(a) \) bounded, i.e., \( ||g'||_{\infty} = sup_{x \in (a,b)}|g'(x)| < +\infty \). Then

\[
\int_a^b g(t)dt - g(x) \leq \left[ \frac{1}{4} \left( \frac{x - a(g)}{b - a} \right) \right] ||g'||_{\infty}
\]

for any \( x \in [a,b] \). The constant \( 1/4 \) is the best possible.

Remark 2.3. For estimation (1.10), assuming \( g \in C[a,b] \cap C^2(a,b) \) with \( ||g'||_{\infty} < +\infty \), we have

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\[
\int_{t_n}^{t_{n+1}} g'(x(s))ds - \mu_n g'(x_n) \leq \frac{\mu_n}{2} \|g''\|_\infty
\]
\[
\int_{t_n}^{t_{n+1}} (t_{n+1} - s)g'(x(s))ds - \mu_n^2 g'(x_n) \leq \frac{\mu_n}{2} \|g''\|_\infty
\]

The system (1.8) introduces a map

\[
\tilde{G} [x_n, p_n, \mu_n] = \begin{pmatrix}
 x_n + p_n \mu_n + \mu_n f(x_n) + \mu_n \int_{t_n}^{t_{n+1}} g(x(s))ds + \int_{t_n}^{t_{n+1}} (t_{n+1} - s)g'(x(s))ds \\
p_n + \mu_n \int_{t_n}^{t_{n+1}} g(x(s))ds + f(x_n) \\
H(\mu_n)
\end{pmatrix}
\]

By assumptions of Theorem 2.1, \( z = (0, 0, \mu_c) \) is a fixed point for \( \tilde{G} \) and the corresponding derivative is

\[
D\tilde{G}(z) = 
\begin{pmatrix}
 1 + \mu_r f'(0) + \mu_r \int_{t_n}^{t_{n+1}} g'(x(s))ds + \int_{t_n}^{t_{n+1}} (t_{n+1} - s)g'(x(s))ds & \mu_r & 0 \\
 f'(0) + \mu_r \int_{t_n}^{t_{n+1}} g'(x(s))ds & 1 & 0 \\
 0 & 0 & 2 - \alpha
\end{pmatrix}
\]

Since the third rows and columns of both \( D\tilde{F}(z) \) and \( D\tilde{G}(z) \) are same, we work only with (3,3)-minors of these matrices.

Let the characteristic polynomials of (3,3)-minors of \( D\tilde{F}(z) \) and \( D\tilde{G}(z) \) be denoted by \( \rho\tilde{F}(\lambda) = \lambda^2 + b\lambda + c \) and \( \rho\tilde{G} = \lambda^2 + b'\lambda + c' \).

**Theorem 2.3.** Let \( g \in \mathcal{C}[a, r] \cap \mathcal{C}^2(a, r) \) with \( \|g''\|_\infty < +\infty \) and suppose either of the following holds:

\[
\|g''\|_\infty < 2 \min \left\{ \frac{\gamma_i - 1}{\mu_i}, \frac{|r|}{2} \right\}, \quad i = 0, 1
\]

where \( r - s \leq \gamma_0 < 1 \) if \( r - s < 1 \) \( 1 < \gamma_1 \leq r - s \) if \( r - s > 1 \). Then for all eigenvalues \( \lambda \) of \( D\tilde{G}(z) \) we have \( |\lambda| > 1 \), i.e., the point \((0, 0, \mu_c)\) is an expansive fixed point for the map \( \tilde{G} \).

If we further assume that

\[
\|g''\|_\infty < 2|1 + x_j^2 g'(0)|, \quad j = 0, 1
\]

then, the dynamic system introduced by the map (2.12) represents the same chaotic behavior as the system (1.11) does.

**Proof.** Let \( \lambda \) be an eigenvalue of \( D\tilde{F}(z) \). It is immediate that \( |\lambda\tilde{F}| > 1 \) if and only if \( r - s \leq \gamma_0 < 1 \) when \( r < 0 \) or \( r - s > \gamma_1 > 0 \) when \( r > 0 \) hold for some \( \gamma_0, \gamma_1 \). Letting \( r' = r + \delta \) and \( s' = s + \delta \) with \( \delta \in [-\|g''\|_\infty /2, \|g''\|_\infty /2] \), the assumptions (2.12) provide a sufficient condition for \( r' > 0 \) and \( r' - s' > 1 \) or \( r' < 0 \) and \( r' - s' < 1 \), which is equivalent to \( |\lambda\tilde{G}| > 1 \), where \( \lambda\tilde{G} \) is an eigenvalue of \( D\tilde{G}(z) \). For the second part, denoting the integral \( \int_{t_n}^{t_{n+1}} (t_{n+1} - s)g'(x(s))ds \) by \( A_j \) and utilizing Eq. (2.10), we obtain

\[
D\tilde{G}(0, 0, x_j) = (2 - \alpha)(1 + A_j), \quad j = 0, 1
\]

where

\[
1 + A_j \in \left(1 + x_j^2 g'(0), 1 + x_j^2 g'(0) + \frac{\|g''\|_\infty}{2}, 1 + x_j^2 g'(0) + \frac{\|g''\|_\infty}{2}\right), \quad j = 0, 1
\]

and a sufficient condition for \( 1 + A_j \neq 0 \) \((j = 0, 1)\) is to satisfy Eq. (2.15).
3 Li–Yorke Chaos in Autonomous Equations on Time Scales

Consider the autonomous dynamic equation

\[ x^h(t) = f(x(t)), \quad t \in \mathbb{T} \subset [0, 1] \]  
(3.1)

with the assumption that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is of class \( C^1 \) and \( \mathbb{T} \) is a time scale. Let \( h : [0, 1] \rightarrow [0, 1] \) be continuously differentiable and satisfies \( h(0) = 0 \).

We denote \( \mathbb{T} \) to be the class of all time scales \( \mathbb{T} \) determined by an initial jump \( \mu_0 \in [0, 1] \), i.e.,

\[ \mathbb{T} = \{ \mathbb{T} \subset [0, 1] | \mathbb{T} = \sigma(\mu_0), \text{for some } \mu_0 \in [0, 1], \min\mathbb{T} = 0 \} \]  
(3.2)

where \( \sigma(\mu_0) \) is the orbit of \( \mu_0 \) created by \( h \). Note that corresponding to \( \mathbb{T} \in \mathbb{T} \), Eq. (3.1) can be represented by

\[ x^h(t) = \left( \int_0^t f(x(s)) \Delta s \right) \sum_{k=0}^{\infty} \delta(\sigma(0) - t) \]  
(3.3)

The class of all initial value problems on \( \mathbb{T} \) is denoted by \( \Psi \), i.e., the class of all \( \mathcal{P} \) (for some \( \mathbb{T} \in \mathbb{T} \)) considered by

\[ \mathcal{P}_T : \left\{ \begin{array}{l}
  x^h(t) = f(x(t)) \\
  x(0) = x_0 \in [a, b], \mu(0) = \mu_0 \in [0, 1] 
\end{array} \right. \]  
(3.4)

The collection of all possible dynamics in \( \Psi \) is characterized by the following map:

\[ F(x, y) = (x + yf(x), h(y)) \]  
(3.5)

We are looking for an \( F \)-invariant \( D \subset \mathbb{R}^2 \) upon which the map \( F \) becomes chaotic in the sense of Li–Yorke. This is equivalent to the existence of an uncountable set \( \mathbb{T} \) (a scrambled set) satisfying the following conditions [21]:

(A) For every \( p, q \in \mathbb{T} \) with \( p \neq q \)

\[ \lim \sup_{n \to \infty \infty} |F^n(p) - F^n(q)| > 0 \]  
(3.6)

and

\[ \lim_{n \to \infty \infty} |F^n(p) - F^n(q)| = 0 \]  
(3.7)

(B) For every \( p \in \mathbb{T} \) and periodic point \( q \in [0, 1] \)

\[ \lim_{n \to \infty \infty} |F^n(p) - F^n(q)| > 0 \]  
(3.8)

Since the properties (A) and (B) are deduced from possessing a period three orbit, then the Li–Yorke chaos is defined as follows:

DEFINITION 3.1. Suppose \( h, I \rightarrow I \) is a continuous function where \( I \) is a compact interval. We say that, \( h \) is chaotic in the sense of Li–Yorke if it possesses a period three orbit.

LEMMA 3.1. Suppose \( f, \lambda : \mathbb{R} \rightarrow \mathbb{R} \) be continuous and \( x \in \mathbb{R} \subset \mathbb{R} \), where \( \mathcal{A} \) is bounded. Assume that

\[ \begin{cases} 
  \lambda f \in \left[ 0, \frac{b - a}{2} \right], & \forall t \in \lambda^{-1} \left( \left[ \frac{a + b}{2}, b \right] \right), \forall \mathcal{A} \\
  \lambda f \in \left[ \frac{a - b}{2}, 0 \right], & \forall t \in \lambda^{-1} \left( \left[ a, \frac{a + b}{2} \right] \right), \forall \mathcal{A} 
\end{cases} \]  
(3.9)

then \( \lambda d + \lambda f \) (\( \forall \mathcal{A} \in \mathcal{A} \)) has a fixed point, where \( d \) is the identity map.

Proof. It is immediate since Eq. (3.9) gives \( (Id + \lambda f)(a, b) \subset [a, b] \).

In view of Lemma 3.1, the following statement will be observed.

THEOREM 3.1. Let \( y_0 \in [0, 1] \) be such that \( h^3(y_0) = y_0 \). Then the system introduced by the map \( F : [a, b] \times [0, 1] \rightarrow [a, b] \times [0, 1] \) is chaotic in the sense of Li–Yorke, provided \( f \) satisfies Eq. (3.9).

Proof. Calculating \( F^3(x, y) \) from Eq. (3.5), equalizing with \( (x, y) \), and making use of Lemma 3.1, we obtain a period three point \( (x_0, y_0) \) which implies chaos.

REMARK 3.1. As we have seen in Secs. 2 and 3, the irregularity in system may originate from a complexity in the dynamic of \( h \). On the other hand, taking \( f \) into account in a specific manner to give it more importance than that of indication in Lemma 3.1 implies the same level of complexity as \( h \) causes. Indeed, assuming the conjugacy between Eq. (3.5) and the map \( G(x, y) = (f(x), h(y)) \), i.e., \( \phi \circ F = G \circ \phi \), for some homeomorphism \( \phi = (\phi_1, \phi_2) \), it will be easy to verify that the

Fig. 2 A schematic view for mapping a block \((x_k, \mu_k)\) to the encrypted block \( F^m(x_k, \mu_k) \) via introducing a new time scale \( \mathbb{T}_k \) with its corresponding \( \mathcal{P}_k \)
For each private key—with any positive error yields to a failure in determination of the length of the magnitude of the length of the kicks pattern is introduced and then a chaotic behavior of the corresponding approximated discretized system caused by the maps \( F \) defined by Eq. (3.5) is chaotic and satisfies Eqs. (3.6)–(3.8), we map \((x_k, \mu_k)\) to \( F_{(x_k, h^{p+1}(\mu_k))} \). Figure 2 represents this mapping schematically in which a primitive data stream including the single blocks \( x_k \) corresponded to \((x_k, \mu_k)\) are converted to a different data stream made of cipher blocks those are corresponded to \( x_k \) proximative) discretized system is concluded. In Sec. 3, a sufficient condition for existence of a period three orbit is stated. As an application, we introduced a symmetric chaotic encryption to a data stream of finite length for which a governing dynamic equation is supposed to be known. In the proposed scheme, the function \( h \), which is the origin of the Li–Yorke chaos, produces sensitivity with respect to an estimation effort, which supports the security of the encrypted data.

It is noteworthy that, in Sec. 2, only the existence of an invariant \( A \) is proved. Also, in Sec. 3, the scrambled set (an uncountable set satisfying Eqs. (3.6)–(3.8)) is unknown. Both problems are beyond the scope of the present manuscript; so, it will be deferred to the author’s next work.

### References


